II. Kinetics, Kinematics of Deformation and Constitutive Relations

2.1 Kinetics of Deformation

2.2 Kinematics of Deformation

2.3 Constitutive Relations

2.1 Kinetics of Deformation

2.2 Kinematics of Deformation

- 2.2.1 Displacement Vector at a Point
- 2.2.2 Deformation of a Deformable Body
- 2.2.3 Strain-Displacement Relationships
- 2.2.4 Analysis of Strain
  - 2.2.4.1 Transformation of Strain Components
  - 2.2.4.2 Principal Strains and Principal Directions
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- 2.3.1 Definitions
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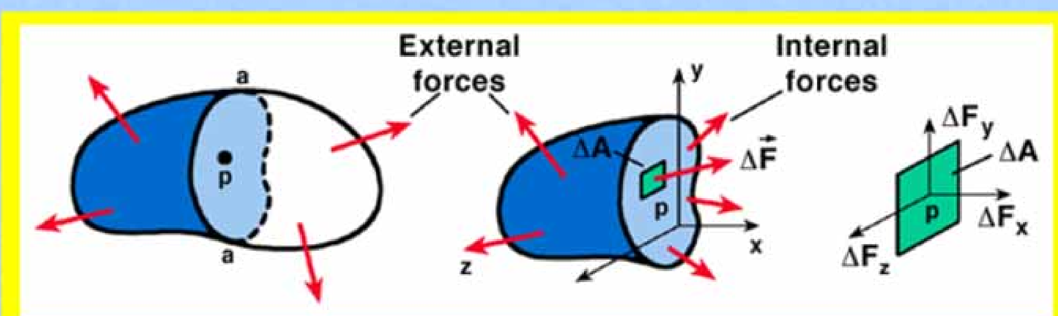
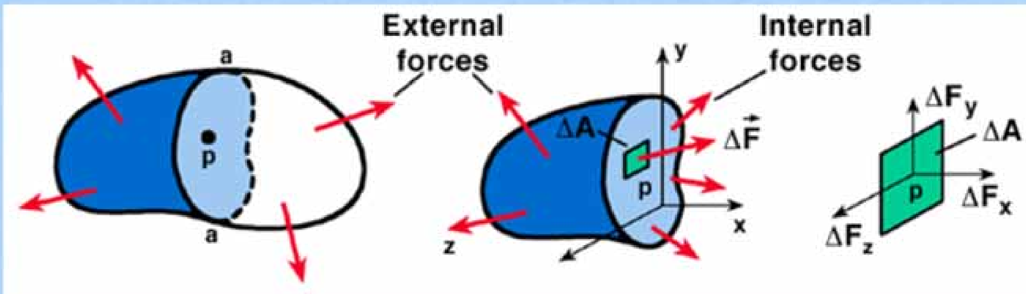
Kinetics, Kinematics of Deformation and *Constitutive Relations*

Kinetics of Deformation

Body in equilibrium under the action of a system of forces (and/or moments)

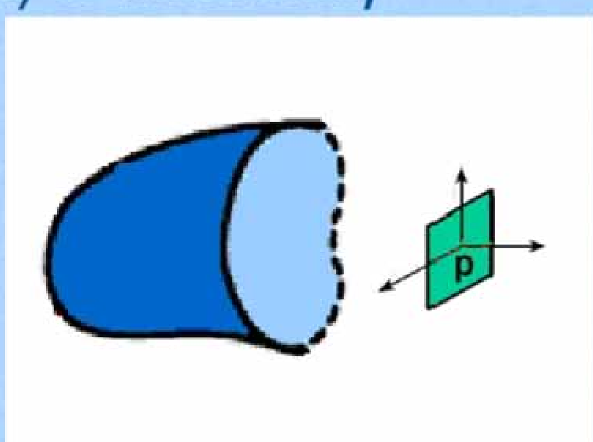


Animation



Kinetics of Deformation

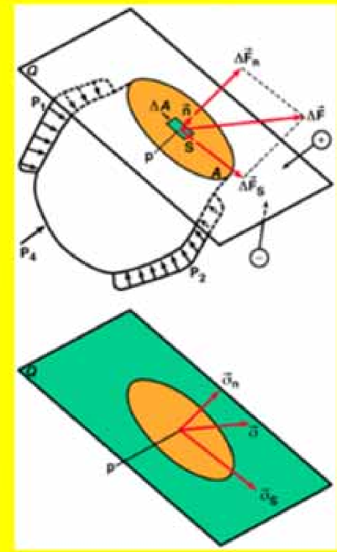
Body in equilibrium under the action of a system of forces (and/or moments)



Kinetics of Deformation

Internal forces are developed within the body.

At any section - internal forces represent the effect of one side on the other, and are in equilibrium with the external forces on the side considered



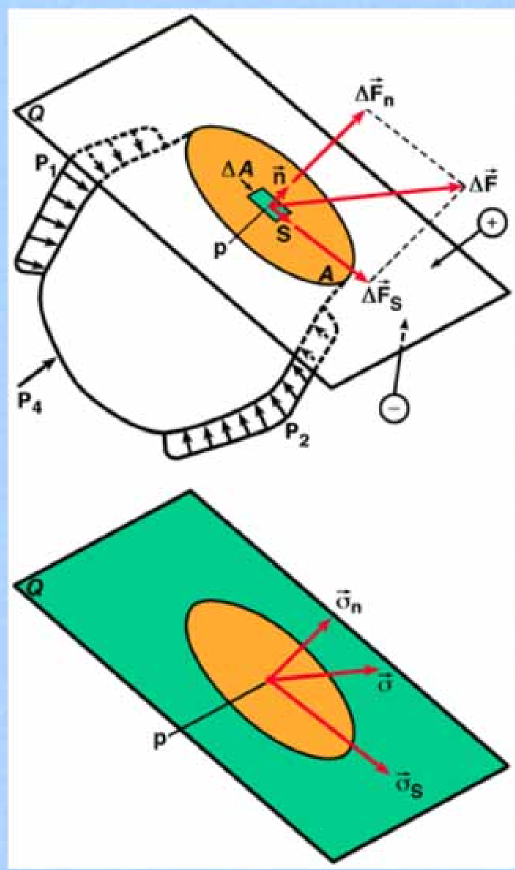
$\Delta\vec{F}$ is the force acting on the area ΔA.

$\Delta\vec{F}_n$ and $\Delta\vec{F}_s$ are normal and tangential components of $\Delta\vec{F}$.

$$\vec{\sigma} = \lim_{\Delta A \to 0} \frac{\Delta\vec{F}}{\Delta A}$$

$$\vec{\sigma}_n = \lim_{\Delta A \to 0} \frac{\Delta\vec{F}_n}{\Delta A}$$

$$\vec{\sigma}_s = \lim_{\Delta A \to 0} \frac{\Delta\vec{F}_s}{\Delta A}$$

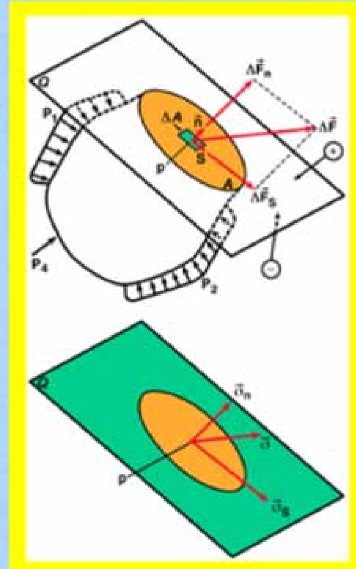


Concept of Stress at a Point

Normal and Tangential Stress Vectors

Stress vector at a point p, associated with the section a-a, is defined as:

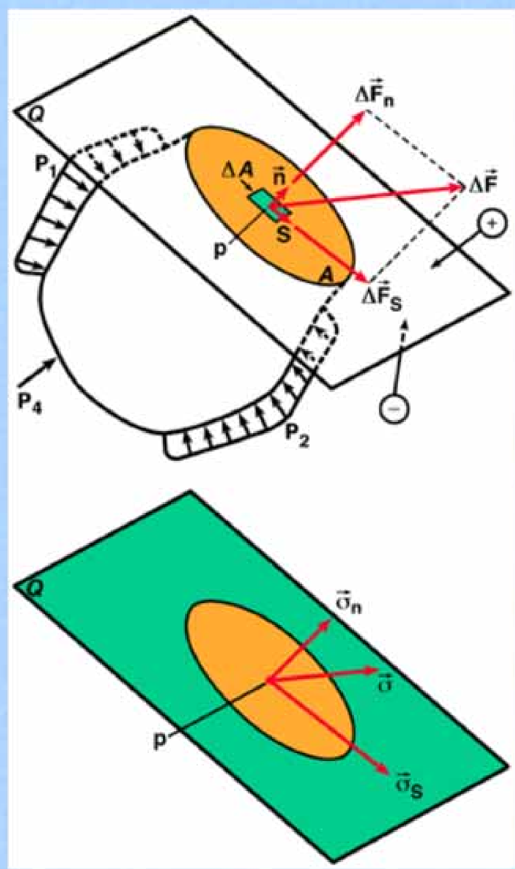
$$\vec{\sigma} = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}}{\Delta A}$$



$$\vec{\sigma} = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}}{\Delta A}$$

$$\vec{\sigma}_n = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}_n}{\Delta A}$$

$$\vec{\sigma}_s = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}_s}{\Delta A}$$



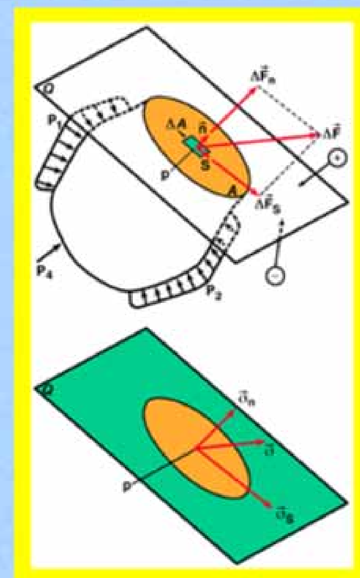
Concept of Stress at a Point

Normal and Tangential Stress Vectors

Normal and tangential (shear) stress vectors at point p, associated with section a-a, are defined as:

$$\vec{\sigma}_n = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}_n}{\Delta A}$$

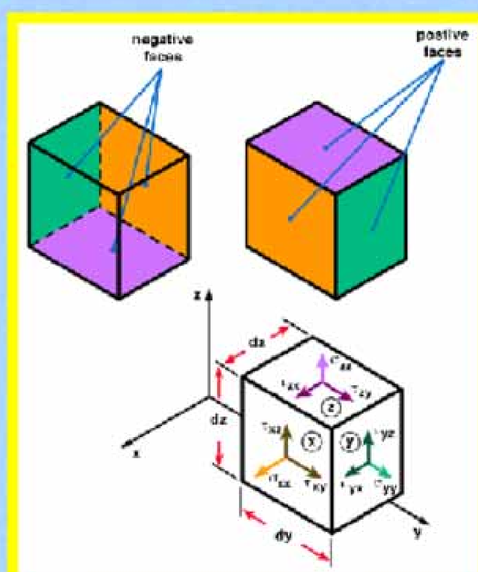
$$\vec{\sigma}_s = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}_s}{\Delta A}$$

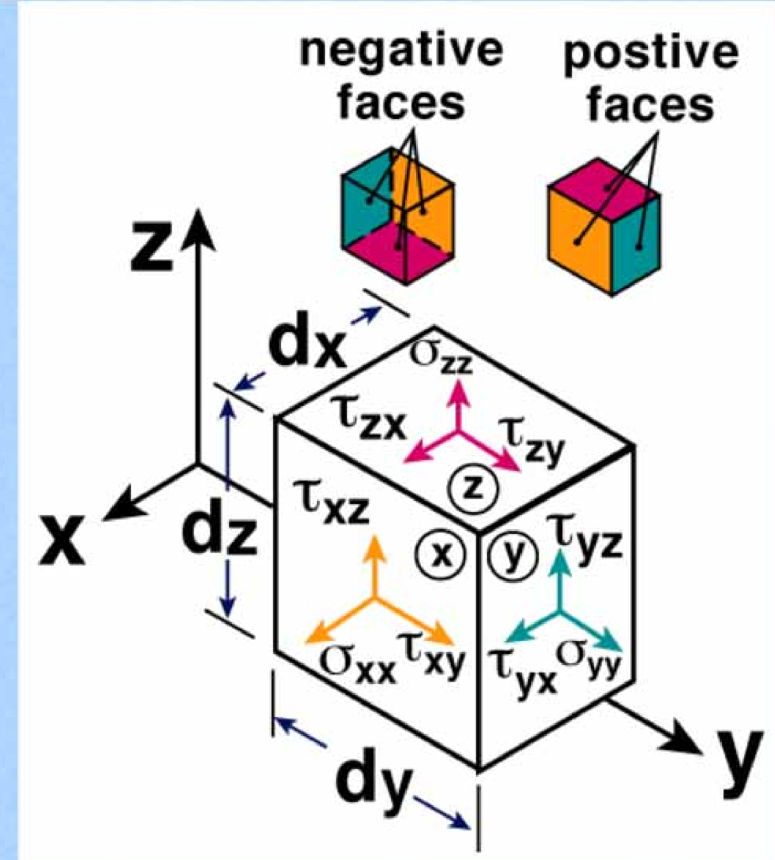


Stress Matrix at a Point

A cubiod with side lengths dx, dy, dz is constructed at the point

Positive faces are defined as those for which the outward normals are in the direction of the positive coordinate axes.





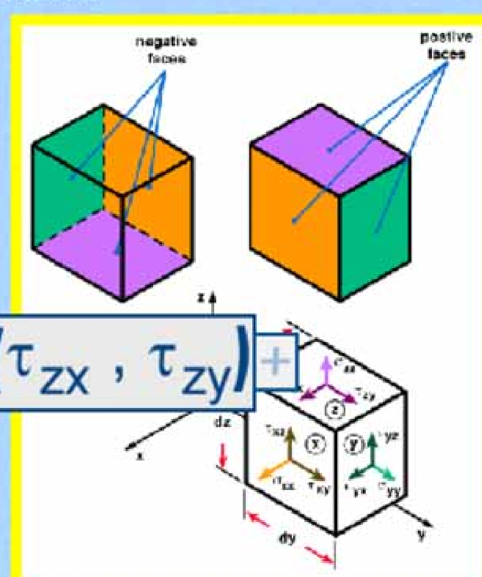
Sign Convention for Stress Components

Positive normal stresses are tensile

$$\sigma_{xx}\ ,\ \sigma_{yy}\ ,\ \sigma_{zz}$$

Positive shear stresses on the positive faces are in the positive coordinate directions

$$(\tau_{xy}\ ,\ \tau_{xz})\ ,\ (\tau_{yx}\ ,\ \tau_{yz})\ ,\ (\tau_{zx}\ ,\ \tau_{zy})\ +$$

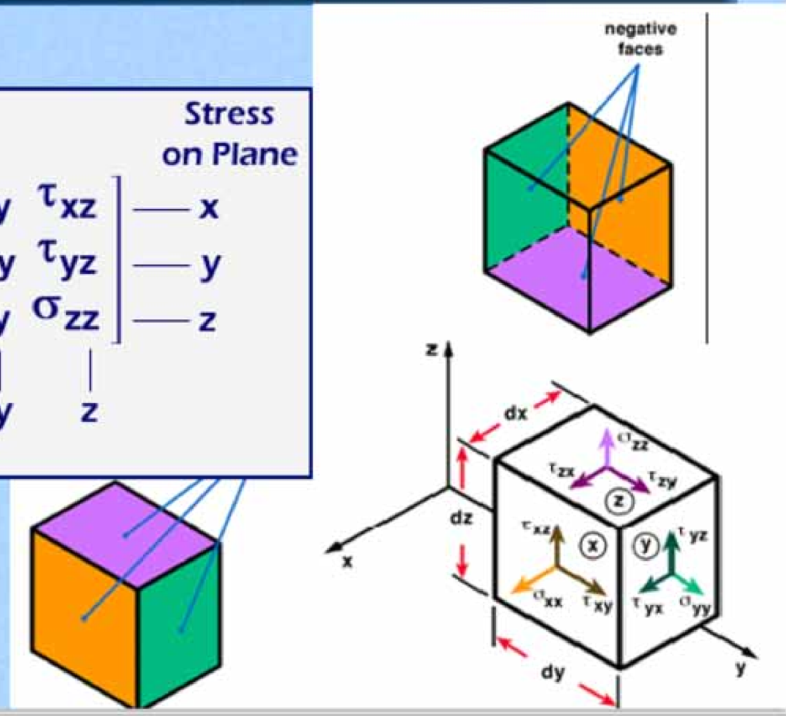


On the negative faces, positive shear stresses are in the negative coordinate directions.

Stress Matrix at a Point

Stress Matrix

$$
\begin{array}{cc}
 & \text{Stress on Plane} \\
[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} & \begin{matrix} x \\ y \\ z \end{matrix} \\
\text{Stress Components in Direction} \quad x \quad y \quad z &
\end{array}
$$



Stress Matrix at a Point

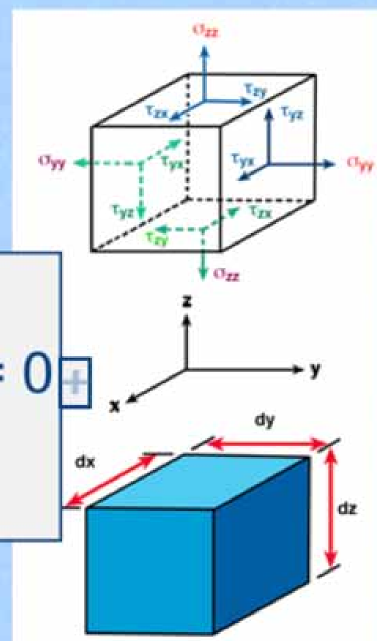
Symmetry of Stress Matrix

Summation of moments about x, y, z leads to:

$$\Sigma M_x = 0$$

$$\left(\tau_{yz}\, dx\, dz\right) dy - \left(\tau_{zy}\, dx\, dy\right) dz = 0$$

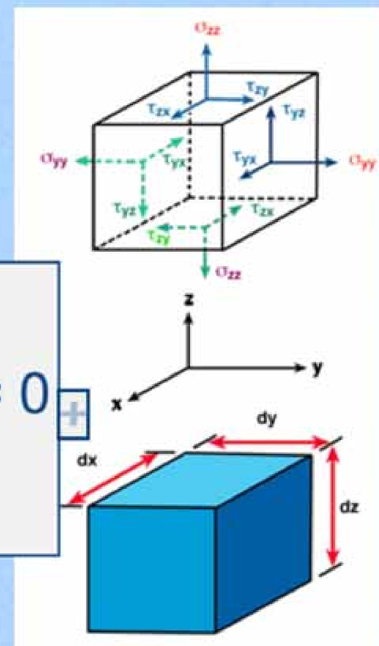
$$\boxed{\tau_{yz} = \tau_{zy}}$$



Stress Matrix at a Point

Symmetry of Stress Matrix

Summation of moments about
x, y, z leads to:



$$\Sigma M_y = 0$$

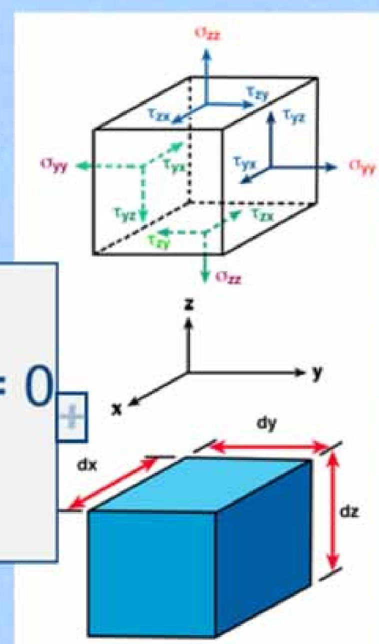
$$\left(\tau_{xz}\, dy\, dz\right) dx - \left(\tau_{zx}\, dx\, dy\right) dz = 0$$

$$\tau_{xz} = \tau_{zx}$$

Stress Matrix at a Point

Symmetry of Stress Matrix

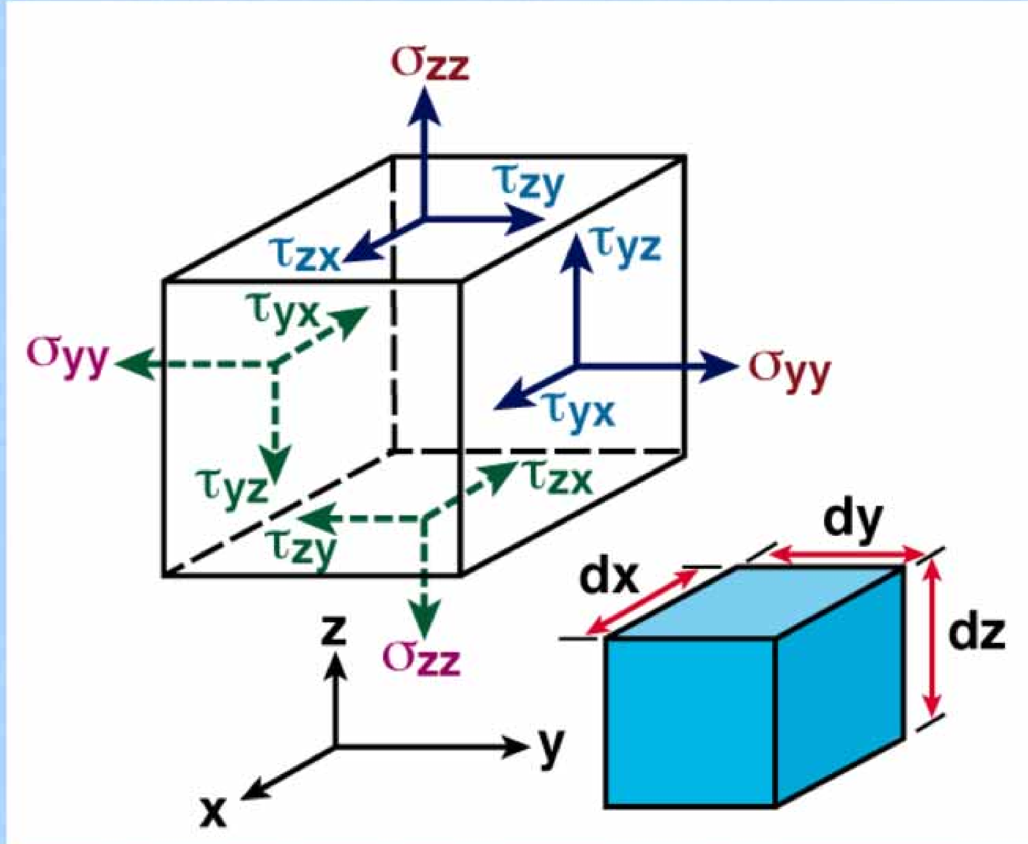
Summation of moments about
x, y, z leads to:



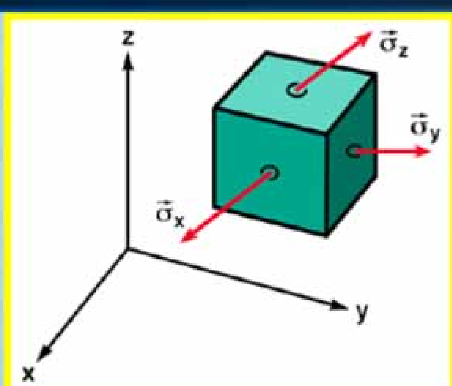
$$\Sigma M_z = 0$$

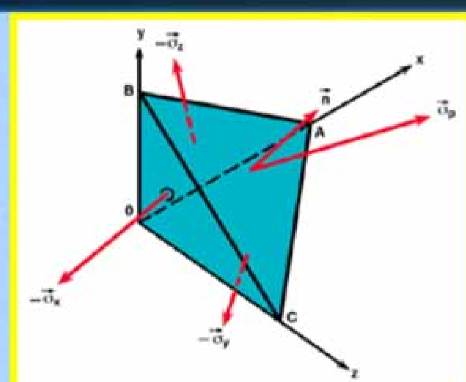
$$\left(\tau_{xy}\, dy\, dz\right) dx - \left(\tau_{yx}\, dx\, dz\right) dy = 0$$

$$\tau_{xy} = \tau_{yx}$$



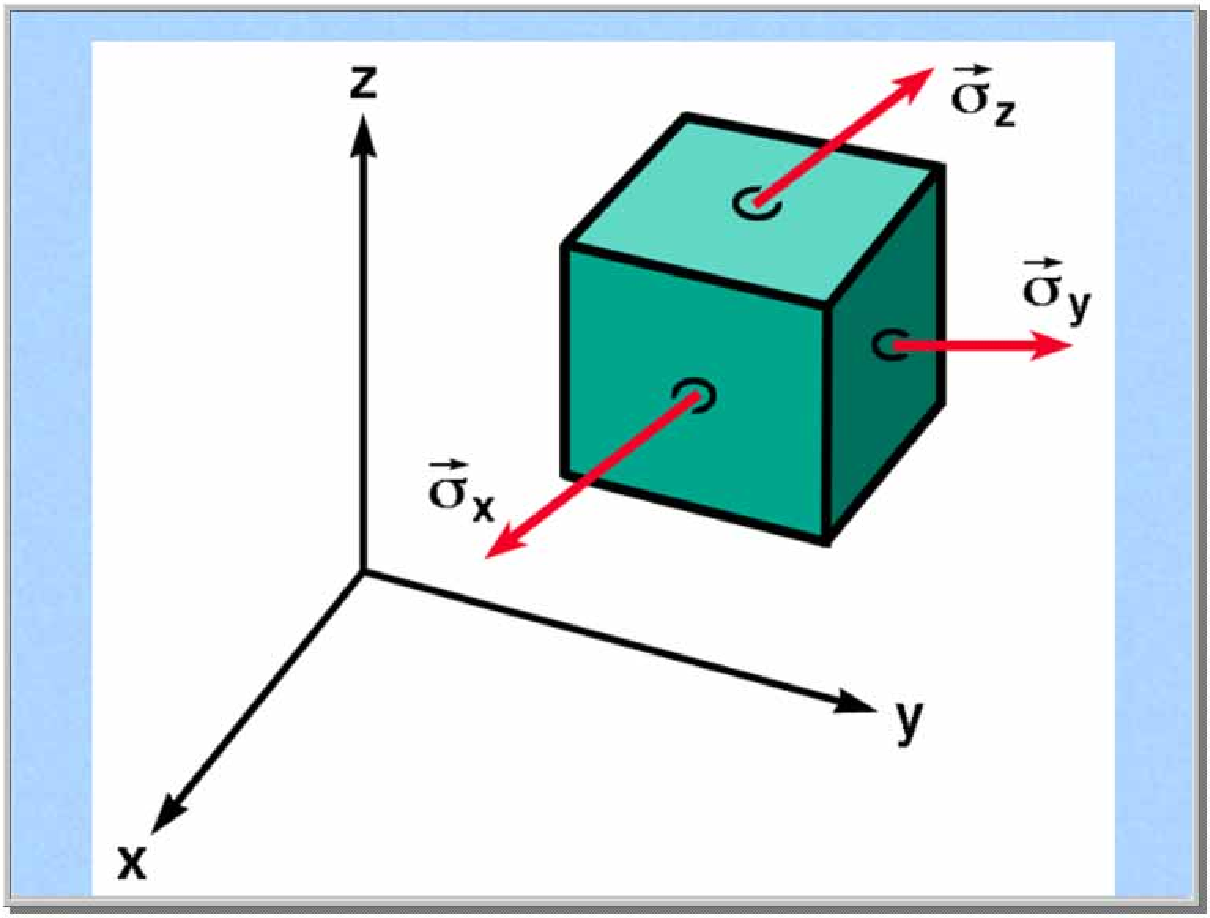
Stress Vector on an Oblique Plane



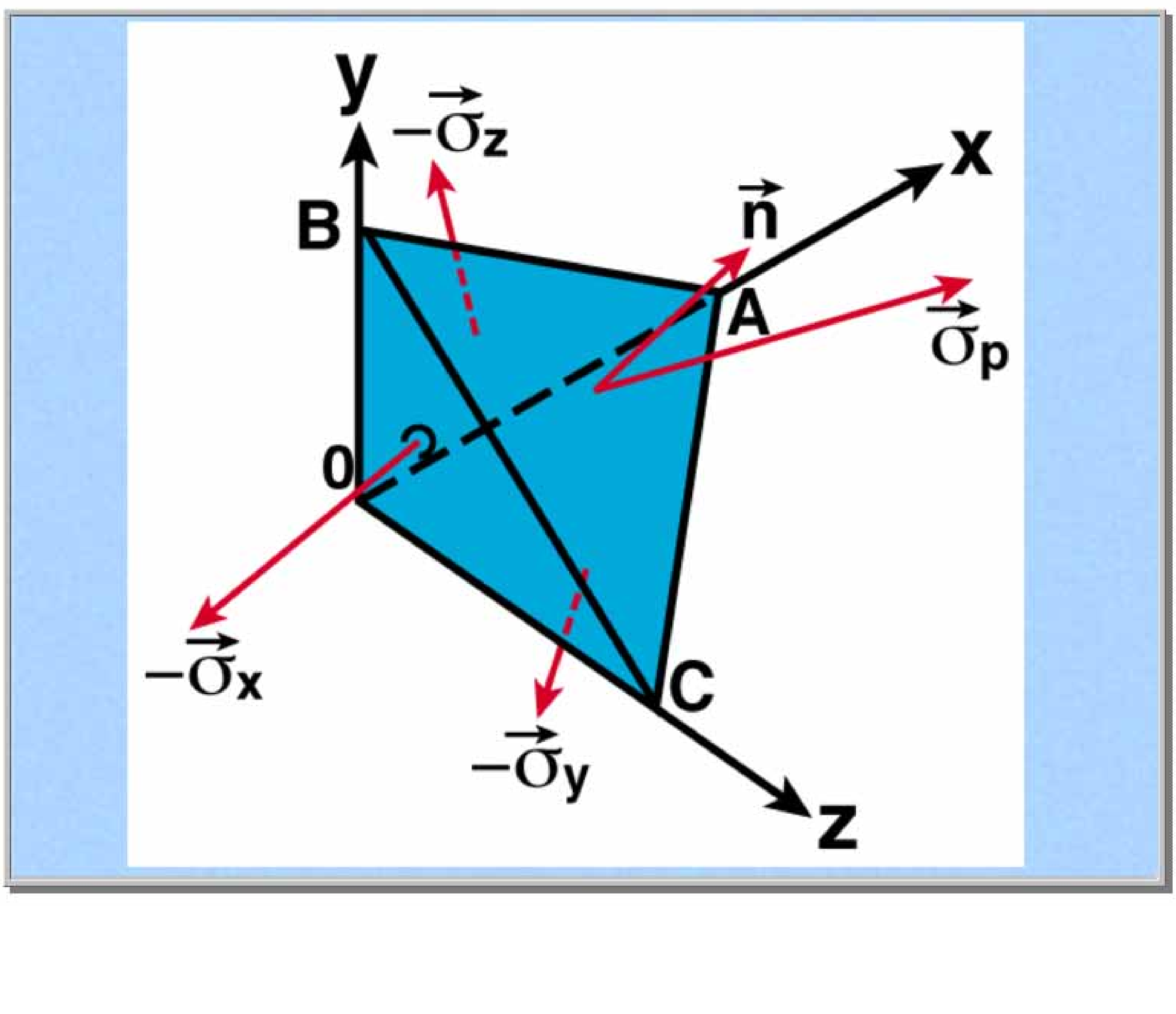


$$\begin{bmatrix} \vec{\sigma}_x \\ \vec{\sigma}_y \\ \vec{\sigma}_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$$

$\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors in x, y, z directions



z
$\vec{\sigma}_z$
$\vec{\sigma}_y$
$\vec{\sigma}_x$
y
x

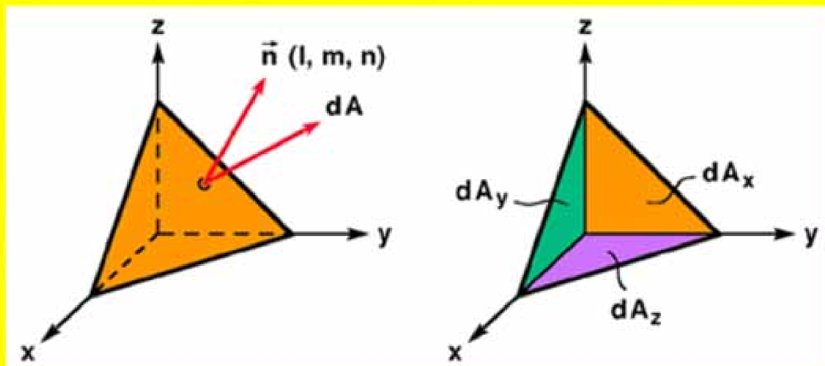


y
$-\vec{\sigma}_z$
x
B
$\vec{n}$
A
$\vec{\sigma}_p$
0
$-\vec{\sigma}_x$
C
$-\vec{\sigma}_y$
z

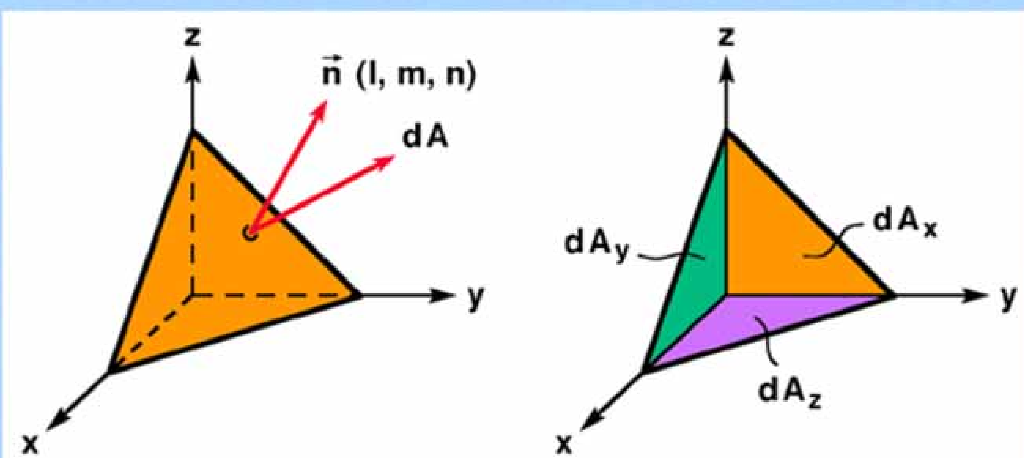
Stress Vector on an Oblique Plane

Stress Vector on Oblique Plane p with unit Normal $\vec{n}$

$$\begin{Bmatrix} dA_x \\ dA_y \\ dA_z \end{Bmatrix} = \begin{Bmatrix} \ell \\ m \\ n \end{Bmatrix} dA$$



Stress Vector on an Oblique Plane



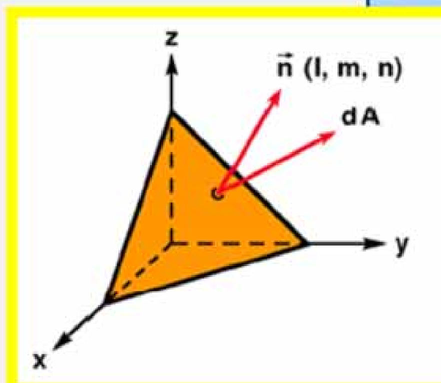
Stress Vector on an Oblique Plane

Equilibrium of Infinitesimal Tetrahedron

$$\vec{\sigma}_p \, dA - \vec{\sigma}_x \, dA_x - \vec{\sigma}_y \, dA_y - \vec{\sigma}_z \, dA_z = 0$$

$$\vec{\sigma}_p = \begin{bmatrix} \vec{\sigma}_x & \vec{\sigma}_y & \vec{\sigma}_z \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix}$$

$$\vec{n} = \begin{bmatrix} \ell & m & n \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$$

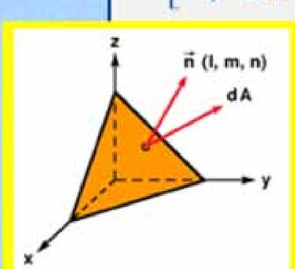


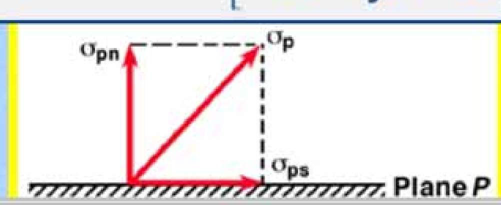
Stress Vector on an Oblique Plane

Normal Stress Components σ_{pn}

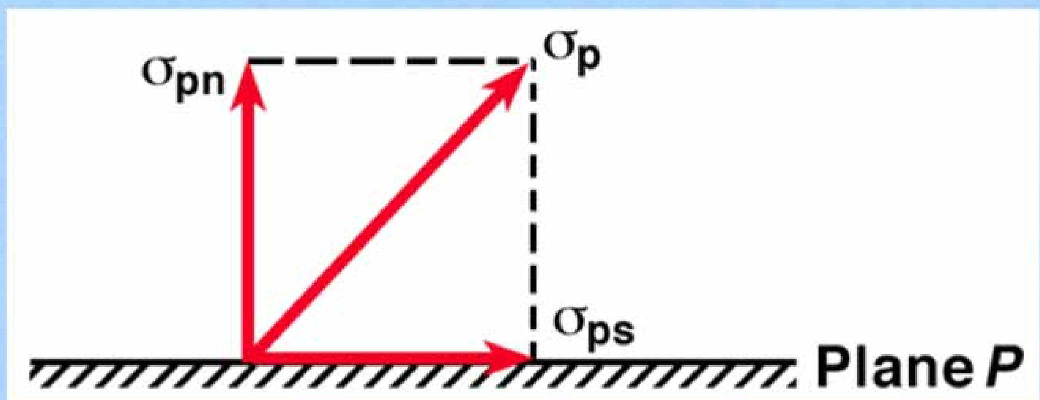
$$\sigma_{pn} = \vec{n} \cdot \vec{\sigma}_p = \begin{bmatrix} \ell & m & n \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} \cdot \begin{bmatrix} \vec{\sigma}_x & \vec{\sigma}_y & \vec{\sigma}_z \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix}$$

$$= \begin{bmatrix} \ell & m & n \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} \cdot \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix}$$

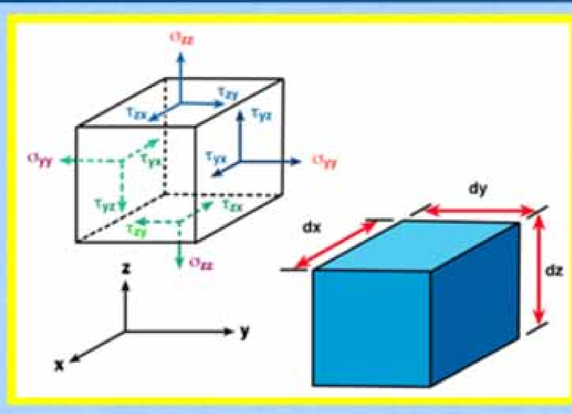


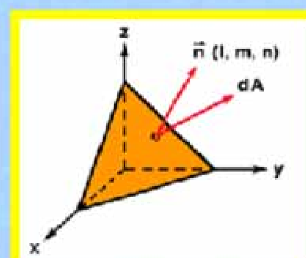


Stress Vector on an Oblique Plane

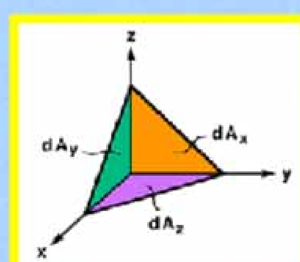


Stress Vector on an Oblique Plane





$$\sigma_{pn} = \begin{bmatrix} \ell & m & n \end{bmatrix} \begin{vmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix}$$

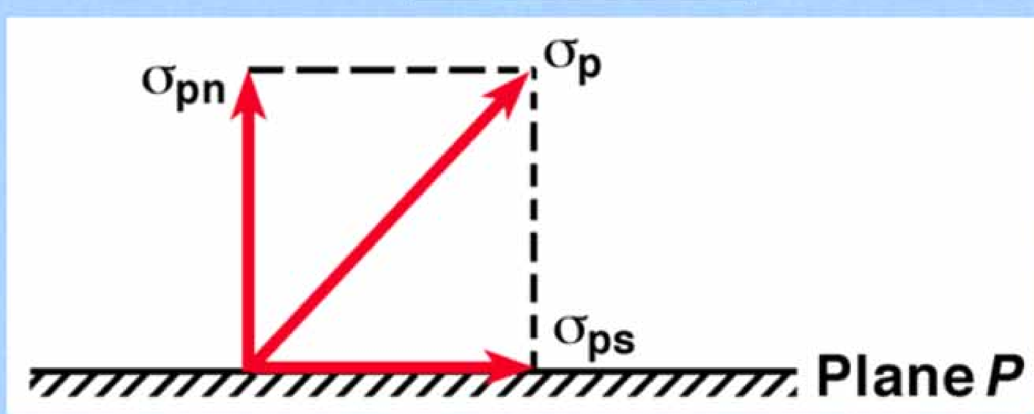


Stress Vector on an Oblique Plane

Shear Stress Component

$$\sigma_{ps} = \sqrt{(\sigma_p)^2 - (\sigma_{pn})^2}$$

where $(\sigma_p)^2 = \vec{\sigma}_p \cdot \vec{\sigma}_p$

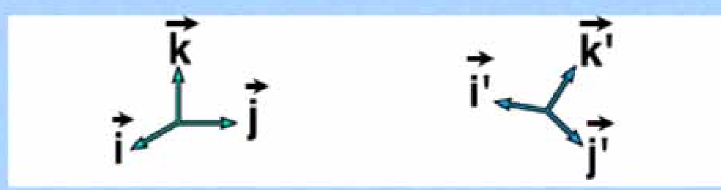


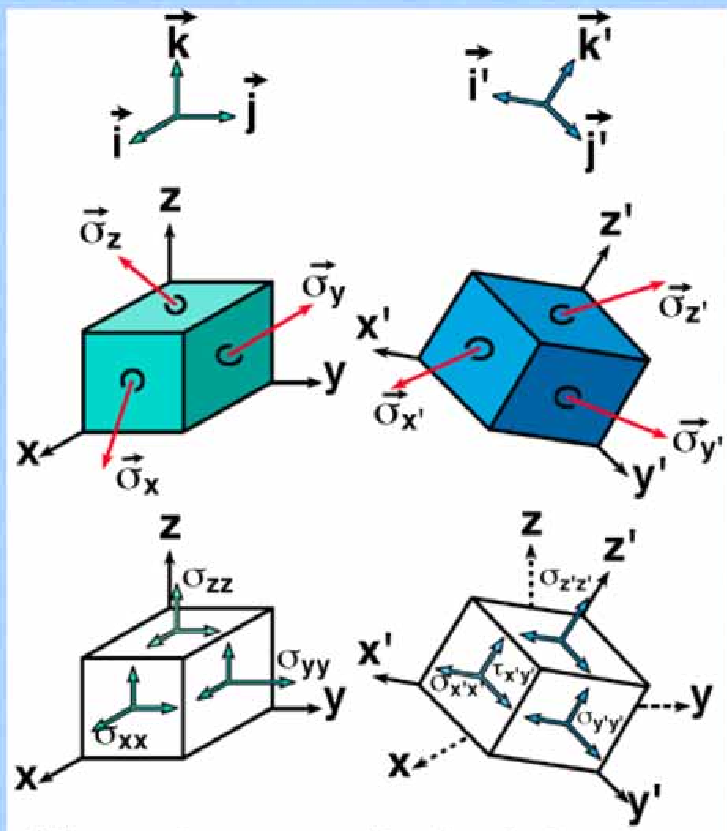
Effect of Transformation of Coordinates on Stress Components

New Coordinate System

Unit vectors $\vec{i}'$, $\vec{j}'$, $\vec{k}'$ are in the direction of the new coordinate x', y', z'.

$$\begin{bmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$$

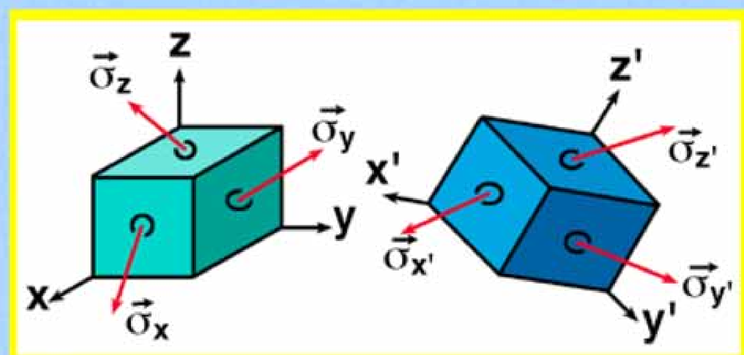




Given stresses   Desired stresses

Effect of Transformation of Coordinates on Stress Components

Stress Vector on the plane x'



$$\vec{\sigma}_{x'} = \begin{bmatrix} \vec{\sigma}_x & \vec{\sigma}_y & \vec{\sigma}_z \end{bmatrix} \begin{bmatrix} \ell_1 \\ m_1 \\ n_1 \end{bmatrix}$$

Effect of Transformation of Coordinates on Stress Components

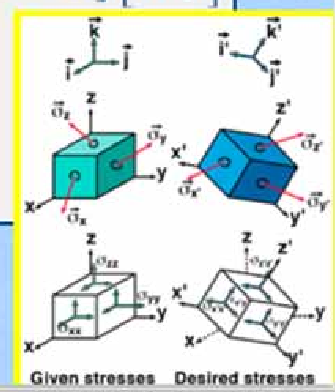
Normal Stress Component

$$\sigma_{x'x'} = \vec{\sigma}_{x'} \cdot \vec{i}'$$

$$= \vec{i}' \cdot \vec{\sigma}_{x'}$$

$$= \begin{bmatrix} \ell_1 & m_1 & n_1 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} \cdot \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \ell_1 \\ m_1 \\ n_1 \end{bmatrix}$$

$$= \begin{bmatrix} \ell_1 & m_1 & n_1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \ell_1 \\ m_1 \\ n_1 \end{bmatrix}$$



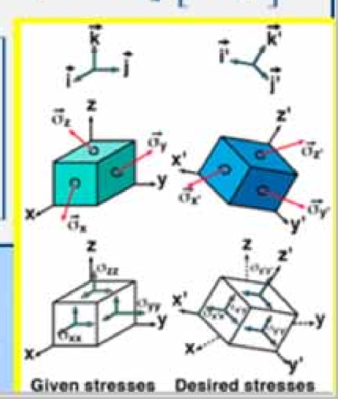
Effect of Transformation of Coordinates on Stress Components

Shear Stress Component

$$\tau_{x'y'} = \vec{j}' \cdot \vec{\sigma}_{x'}$$

$$= \begin{bmatrix} \ell_2 & m_2 & n_2 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} \cdot \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \ell_1 \\ m_1 \\ n_1 \end{bmatrix}$$

$$= \begin{bmatrix} \ell_2 & m_2 & n_2 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \ell_1 \\ m_1 \\ n_1 \end{bmatrix}$$

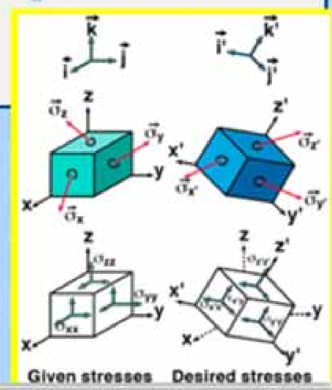


Effect of Transformation of Coordinates on Stress Components

Stress Components at a Point Referred to x', y', z' Coordinate Systems

$$\begin{bmatrix} \sigma_{x'x'} & \tau_{y'x'} & \tau_{z'x'} \\ \tau_{x'y'} & \sigma_{y'y'} & \tau_{z'y'} \\ \tau_{x'z'} & \tau_{y'z'} & \sigma_{z'z'} \end{bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

or $[\sigma'] = [T]\,[\sigma]\,[T]^t$

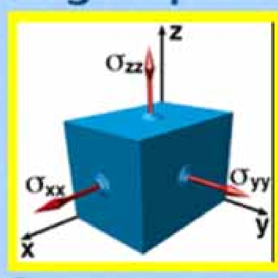


Special States of Stress

Three Dimensional

Principal Stresses Normal stresses acting on planes, on which shearing stresses are zero

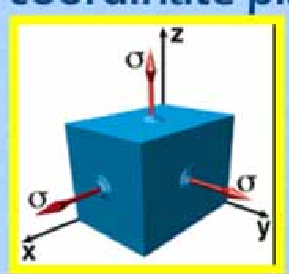
$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \cdot & 0 \\ \cdot & \sigma_{yy} & \cdot \\ 0 & \cdot & \sigma_{zz} \end{bmatrix}$$

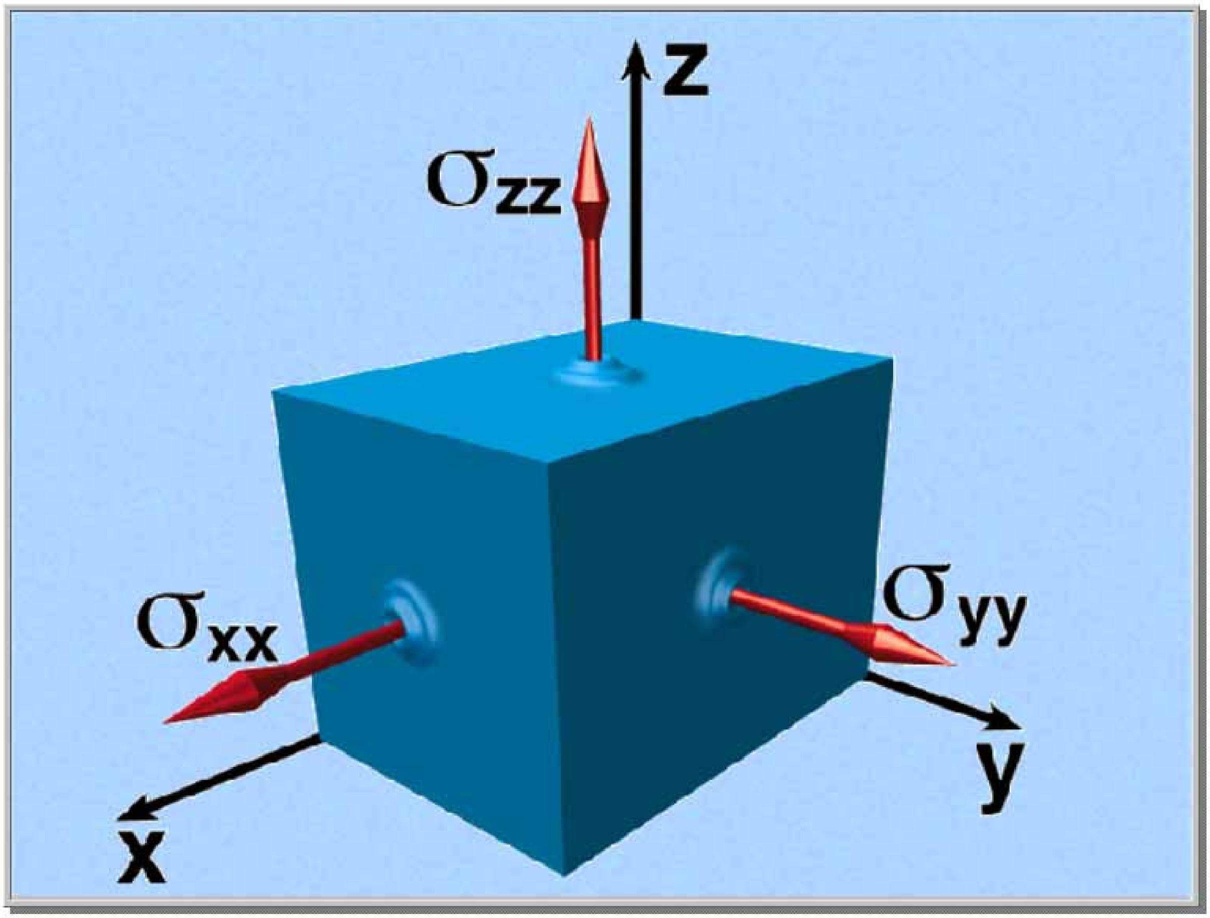


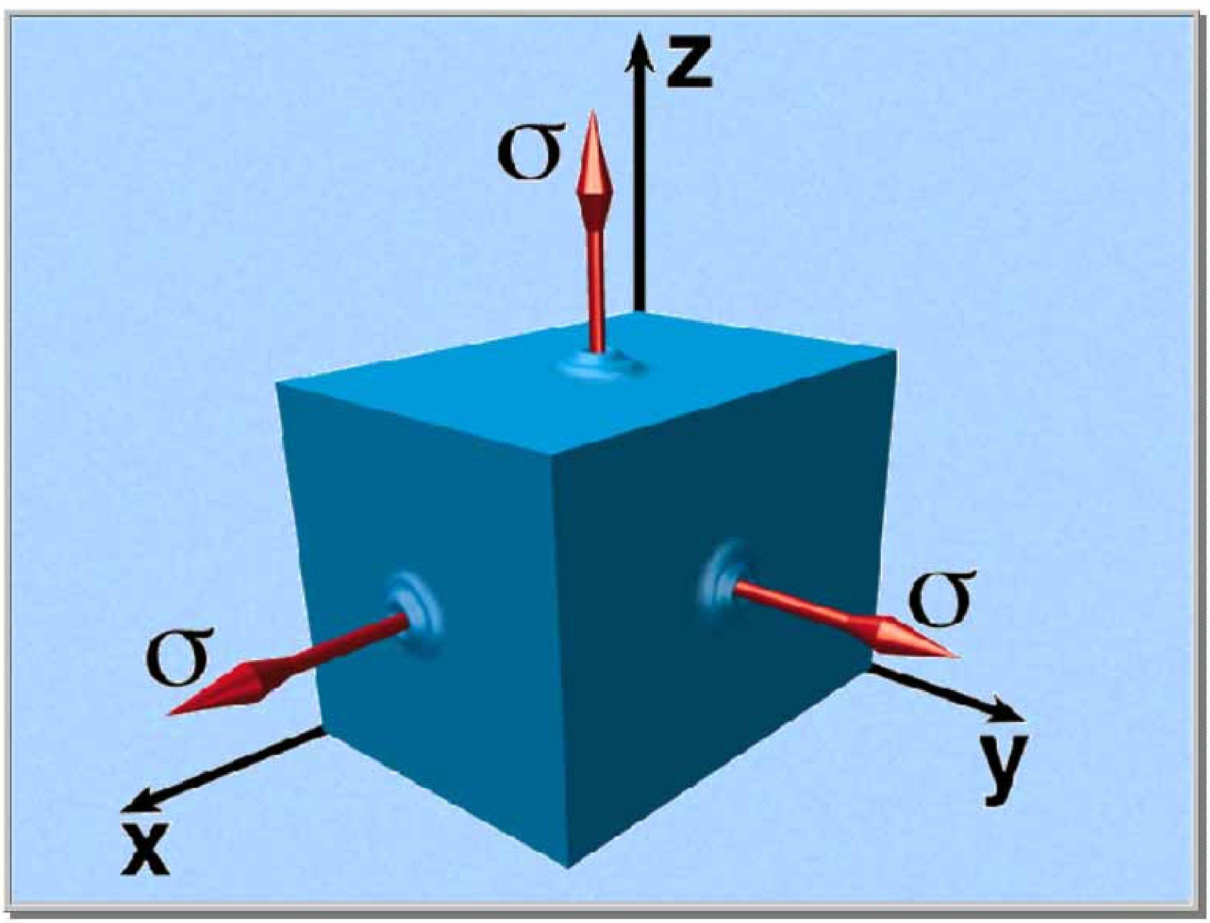
Spherical, Volumetric or Dilatational Stresses

Equal principal stresses on the three coordinate planes

$$[\sigma] = \begin{bmatrix} \sigma & \cdot & 0 \\ \cdot & \sigma & \cdot \\ 0 & \cdot & \sigma \end{bmatrix}$$




z
σzz
σxx
σyy
x
y

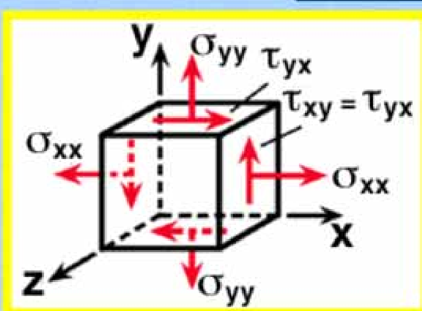

z
σ
σ
σ
x
y

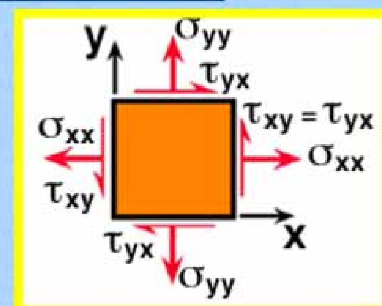
Special States of Stress

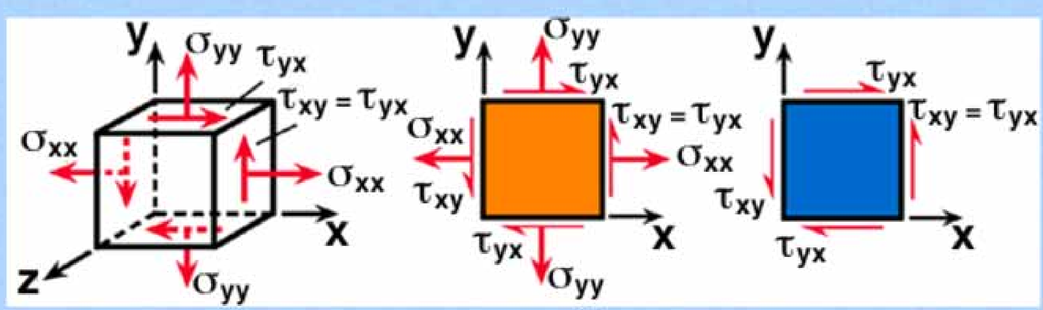
Two-Dimensional (Plane Stresses)

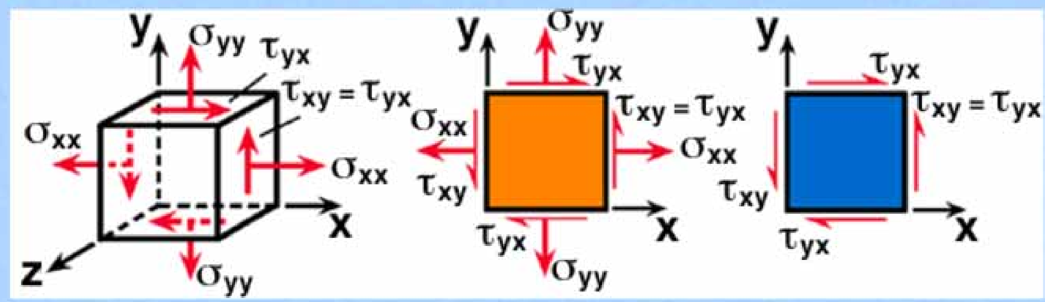
All nonzero stress components are in two coordinate directions only; example, stress state in plane xy

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$









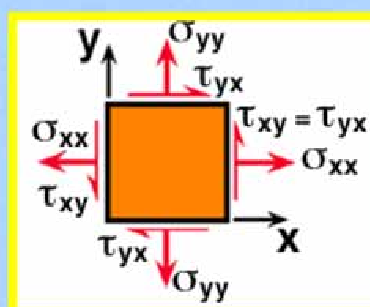
Special States of Stress

Two-Dimensional (Plane Stresses)

Pure Shear

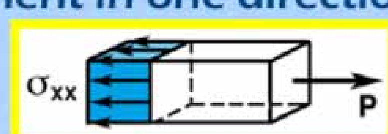
All nonzero stress components are shear stresses in one plane (e.g., x-y plane)

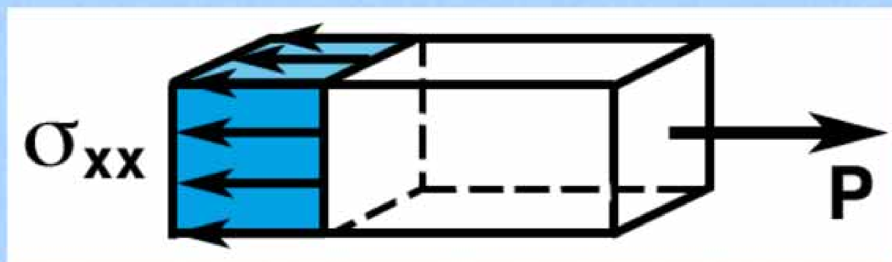
$$[\sigma] = \begin{bmatrix} 0 & \tau_{xy} & 0 \\ \tau_{yx} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Uniaxial Stress

Only the normal stress component in one direction is nonzero



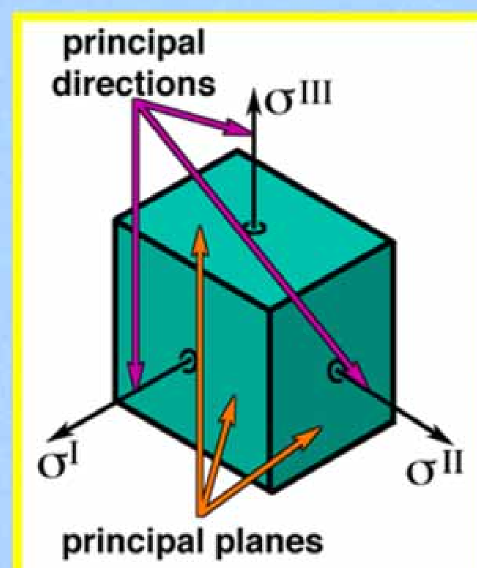


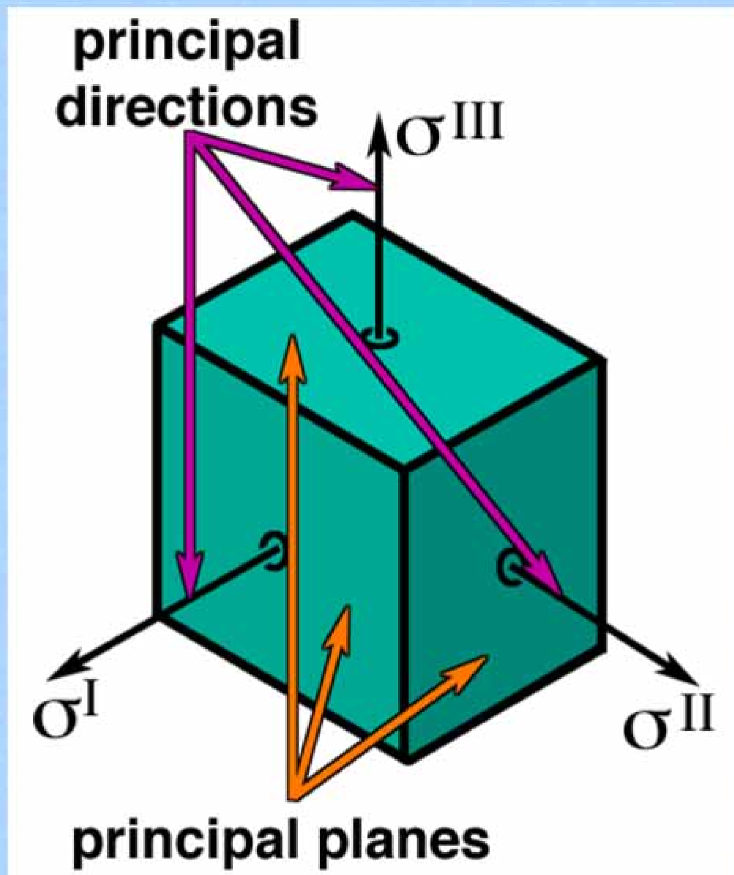
Principal Planes, Principal Stresses and Principal Directions

Principal planes are planes on which the shear stresses vanish.

Principal stresses are normal stresses acting on principal planes.

Principal directions are the directions of principal stresses (mutually orthogonal).



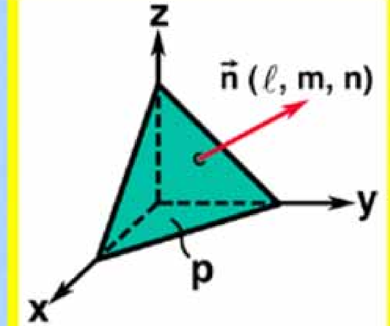


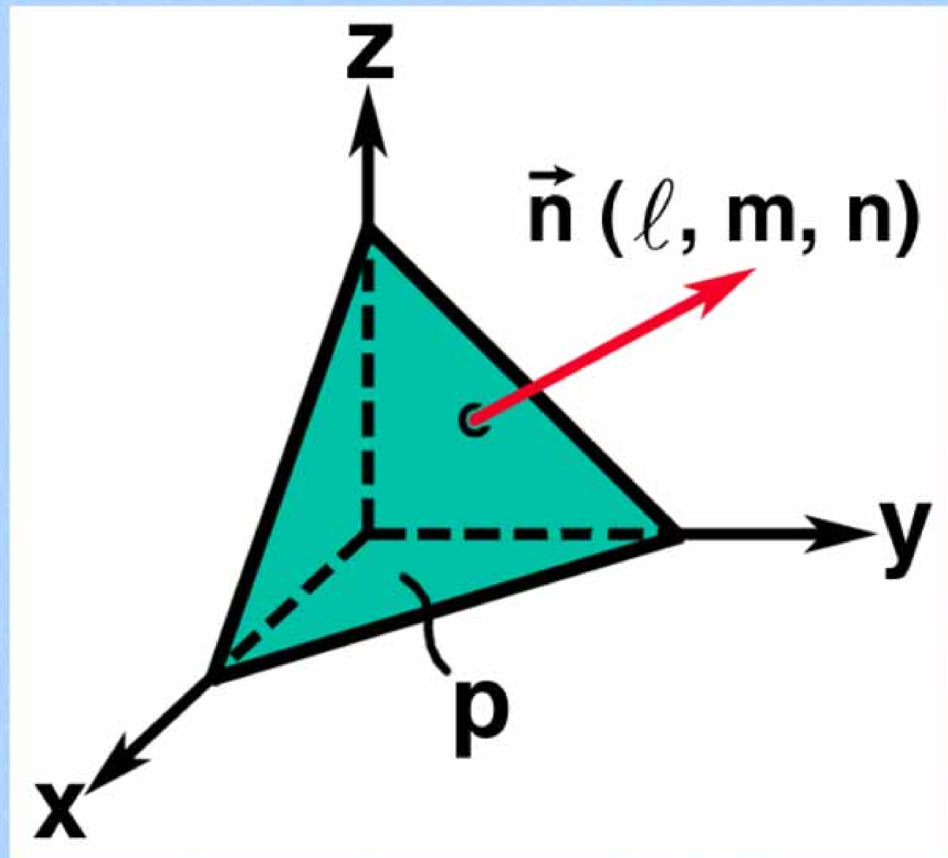
Principal Planes, Principal Stresses and Principal Directions

Determination of principal stresses

Let p be a principal plane whose unit outward normal is $\vec{n}$.

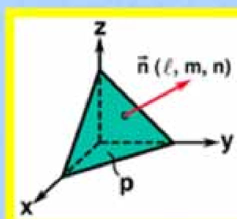
$$\vec{n} = [\ \ell \quad m \quad n\ ] \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$$



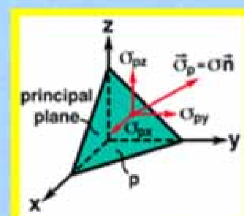


Principal Planes, Principal Stresses and Principal Directions

$$\vec{\sigma}_p = \sigma\,\vec{n} = \begin{bmatrix}\sigma_{px} & \sigma_{py} & \sigma_{pz}\end{bmatrix}\begin{bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{bmatrix}$$

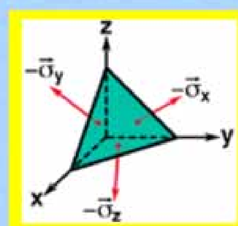


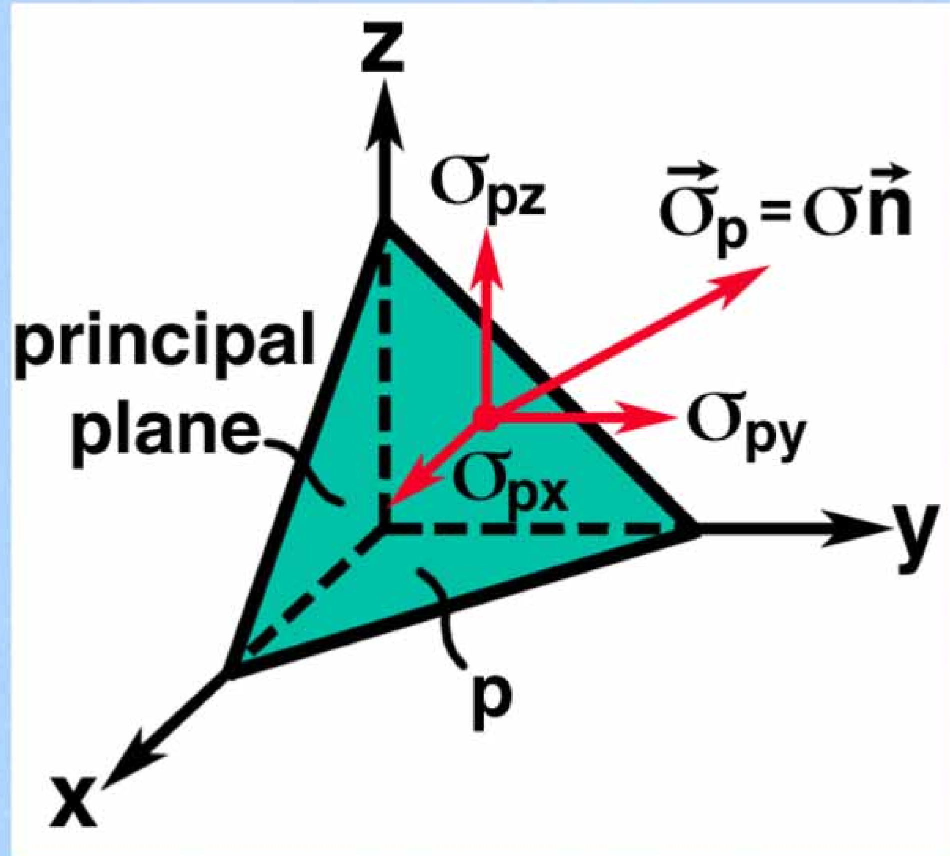
where σ = magnitude of principal stress on the principal plane p.



σ_{px} , σ_{py} , σ_{pz} are the projections of $\overrightarrow{\sigma_p}$ on the coordinate directions, and are given by:

$$\begin{Bmatrix}\sigma_{px}\\ \sigma_{py}\\ \sigma_{pz}\end{Bmatrix} = \sigma\begin{Bmatrix}\ell\\ m\\ n\end{Bmatrix}$$



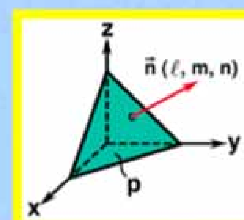


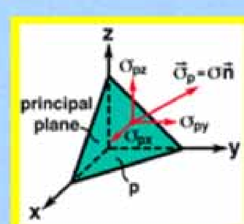
Principal Planes, Principal Stresses and Principal Directions

where σ = magnitude of principal stress on the principal plane p.

σ_{px} , σ_{py} , σ_{pz} are the projections of $\vec{\sigma}_p$ on the coordinate directions, and are given by:

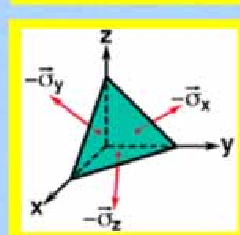
$$\begin{Bmatrix} \sigma_{px} \\ \sigma_{py} \\ \sigma_{pz} \end{Bmatrix} = \sigma \begin{Bmatrix} \ell \\ m \\ n \end{Bmatrix}$$





If the relationship:

$$\vec{\sigma}_p = \begin{bmatrix} \vec{\sigma}_x & \vec{\sigma}_y & \vec{\sigma}_z \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix}$$



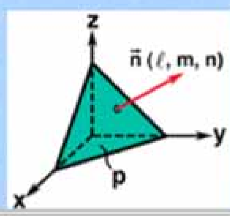
is used then

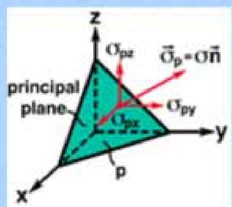
Principal Planes, Principal Stresses and Principal Directions

If the relationship:

$$\vec{\sigma}_p = \begin{bmatrix} \vec{\sigma}_x & \vec{\sigma}_y & \vec{\sigma}_z \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix}$$

is used then



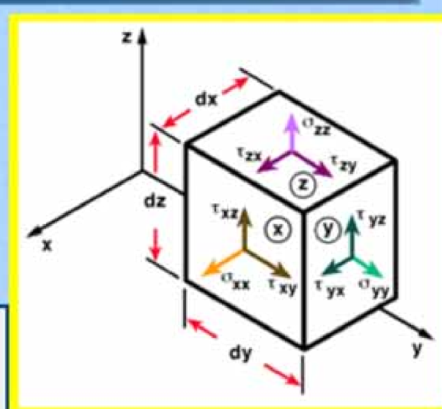


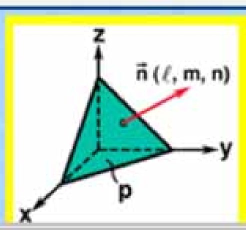
Principal Planes, Principal Stresses and Principal Directions

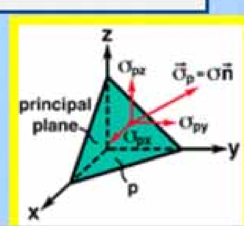
$$\begin{vmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = \sigma \begin{bmatrix} \ell \\ m \\ n \end{bmatrix}$$

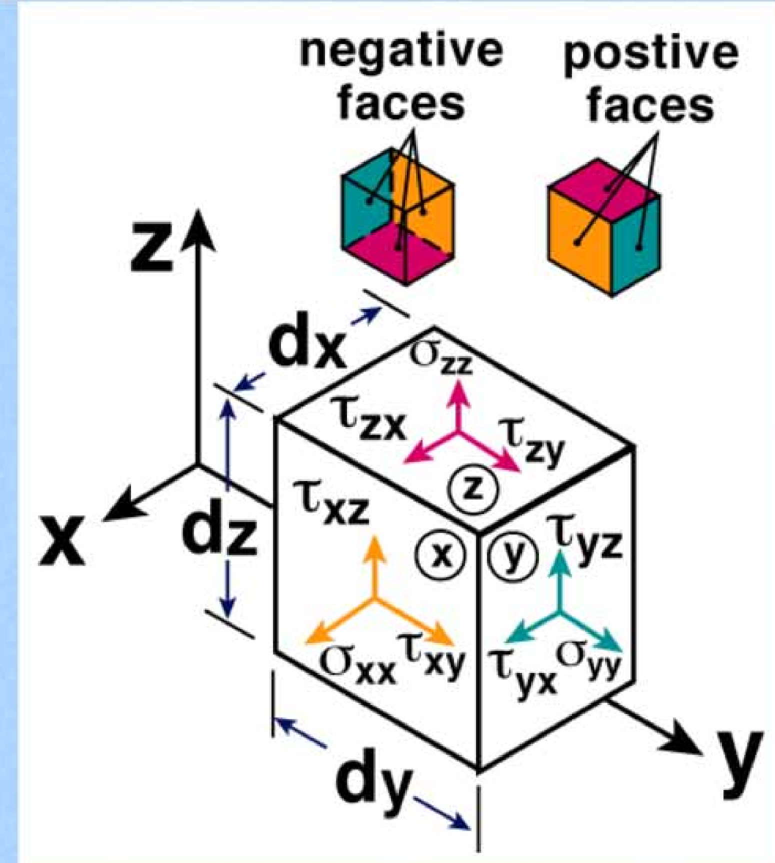
or

$$\begin{vmatrix} \sigma_{xx}-\sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy}-\sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz}-\sigma \end{vmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$







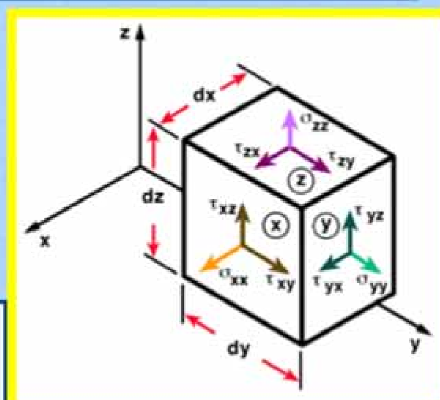


Principal Planes, Principal Stresses and Principal Directions

$$\begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = \sigma \begin{bmatrix} \ell \\ m \\ n \end{bmatrix}$$

or

$$\begin{bmatrix} \sigma_{xx}-\sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy}-\sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz}-\sigma \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} = 0$$



Three linear homogeneous simultaneous algebraic equations in ℓ, m, n - which is an algebraic eigenvalue problem.

Principal Planes, Principal Stresses and Principal Directions

Since

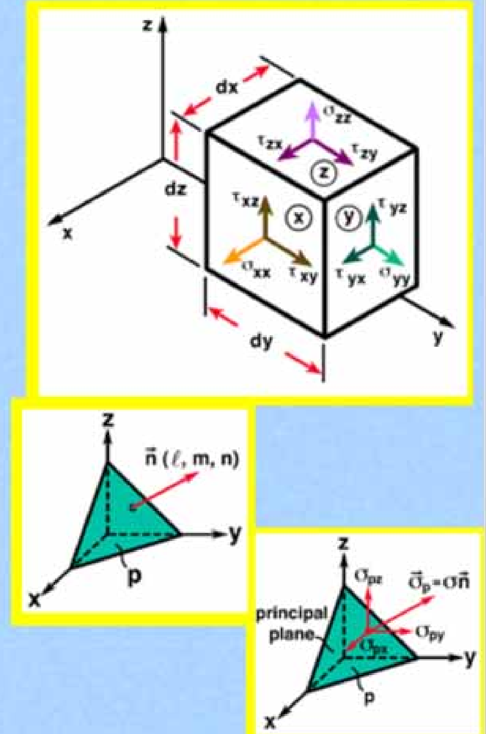
$$\ell^2 + m^2 + n^2 = 1$$

Therefore, the trivial solution $\ell = m = n = 0$ is not possible.

and

$$\det.\begin{vmatrix} \sigma_{xx}-\sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy}-\sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz}-\sigma \end{vmatrix} = 0$$

or, expanding the determinant



Principal Planes, Principal Stresses and Principal Directions

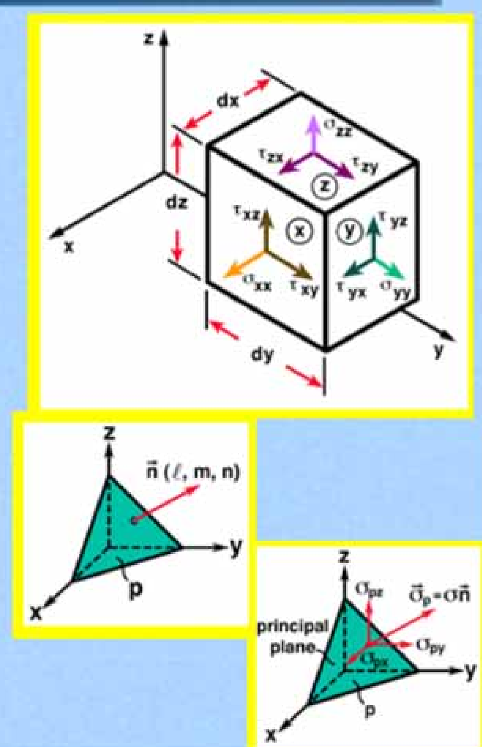
$$\det.\begin{vmatrix} \sigma_{xx}-\sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy}-\sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz}-\sigma \end{vmatrix} = 0$$

or, expanding the determinant

$$-\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0$$

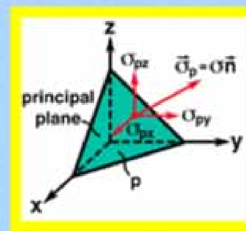
where

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

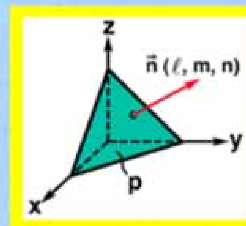


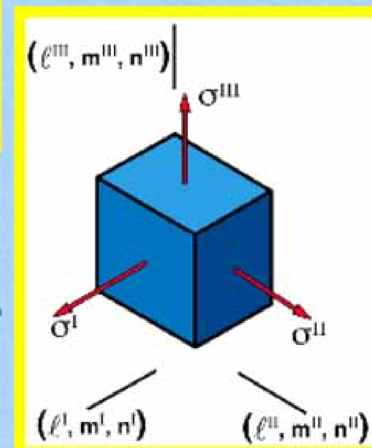
Principal Planes, Principal Stresses and Principal Directions

$$I_2 = \begin{vmatrix} \sigma_{xx} & \tau_{yx} \\ \tau_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \tau_{zx} \\ \tau_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \tau_{zy} \\ \tau_{yz} & \sigma_{zz} \end{vmatrix}$$



$$I_3 = \begin{vmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix}$$

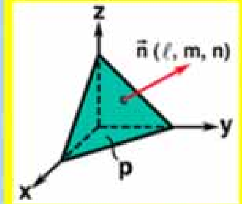


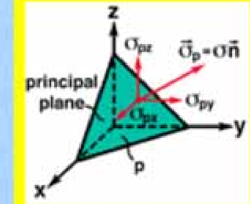


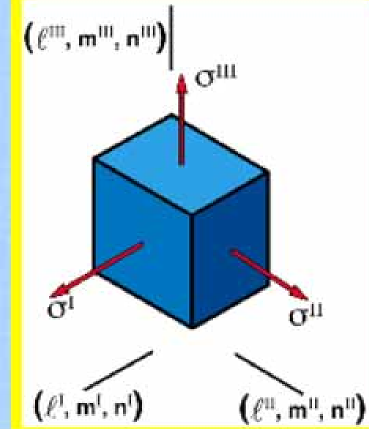
- The quantities I_1, I_2, I_3 do not change with coordinate transformations. They are called stress invariants.

Principal Planes, Principal Stresses and Principal Directions

- The quantities I_1, I_2, I_3 do not change with coordinate transformations. They are called stress invariants.
- The three roots of the cubic equation are the magnitudes of the principal stresses $\sigma^{I}, \sigma^{II}, \sigma^{III}$.







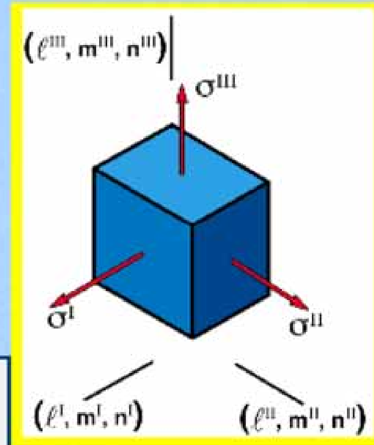
Principal Planes, Principal Stresses and Principal Directions

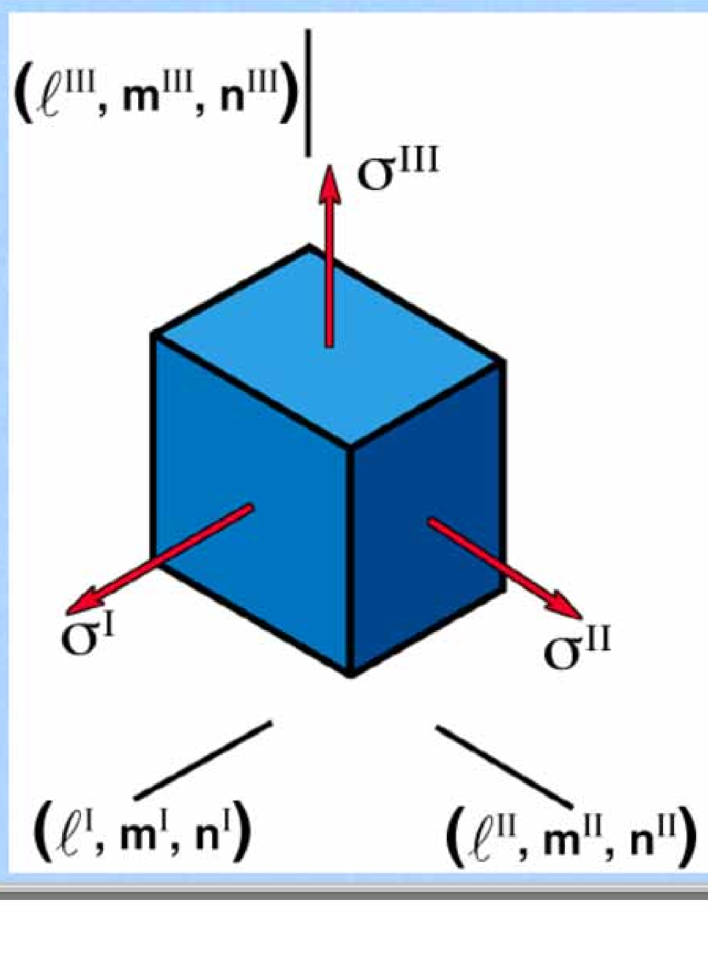
- The three principal directions are obtained by successively replacing σ in the eigenvalue problem by σ^{I}, σ^{II} and σ^{III}, and using the relationship $\ell^2 + m^2 + n^2 = 1$.

$$\boxed{\sigma^{I} \rightarrow \left(\ell^{I}, m^{I}, n^{I}\right)}$$

$$\boxed{\sigma^{II} \rightarrow \left(\ell^{II}, m^{II}, n^{II}\right)}$$

$$\boxed{\sigma^{III} \rightarrow \left(\ell^{III}, m^{III}, n^{III}\right)}$$





Maximum Shear Stresses

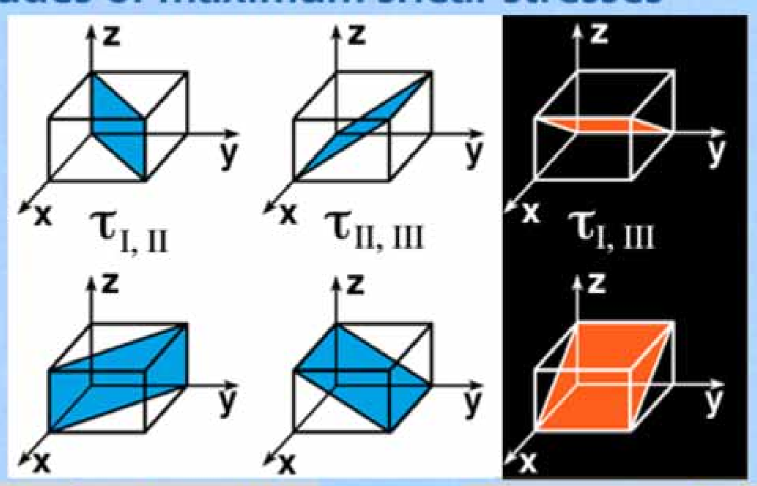
Maximum shear stresses occur on the planes bisecting the angles between the principal planes.

If the principal stresses σ^I, σ^{II}, σ^{III} are in the direction of the x, y, z axes, the planes of maximum shear stresses are such that:

	$\tau^{I,II}$	$\tau^{II,III}$	$\tau^{I,III}$
ℓ	$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$
m	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0
n	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$

Maximum Shear Stresses

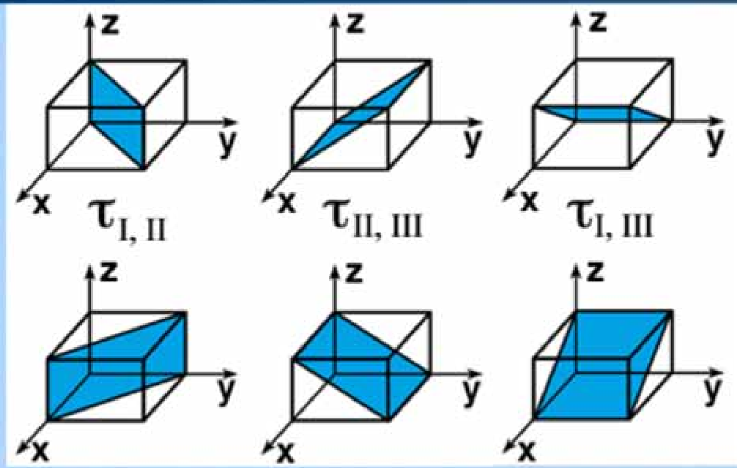
Magnitudes of maximum shear stresses



$$\tau_{I,II} = \pm\frac{1}{2}\left(\sigma^{I} - \sigma^{II}\right) \qquad \tau_{I,III} = \pm\frac{1}{2}\left(\sigma^{III} - \sigma^{I}\right)$$

$$\tau_{II,III} = \pm\frac{1}{2}\left(\sigma^{II} - \sigma^{III}\right)$$

Maximum Shear Stresses



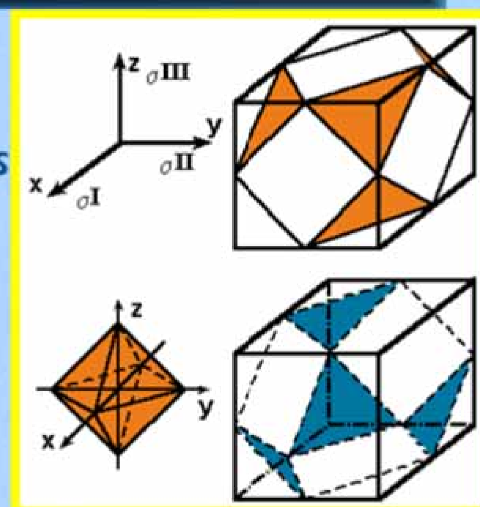
The magnitude of normal stresses acting on the same planes are:

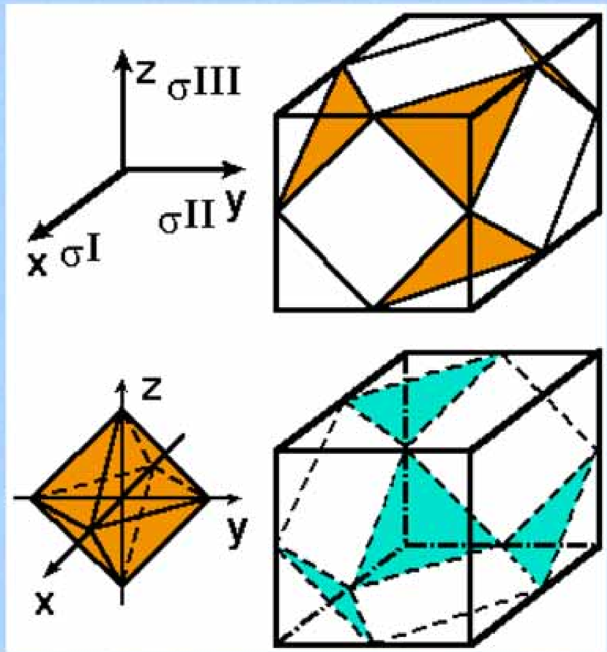
$\frac{1}{2}(\sigma^{I}+\sigma^{II})$ $\frac{1}{2}(\sigma^{II}+\sigma^{III})$ $\frac{1}{2}(\sigma^{III}+\sigma^{I})$

Octahedral Planes and Octahedral Stresses

<u>Octahedral planes</u>
are planes which are equally inclined to the principal planes
The direction cosines of the normals to these planes (relative to the principal axes) are given by:

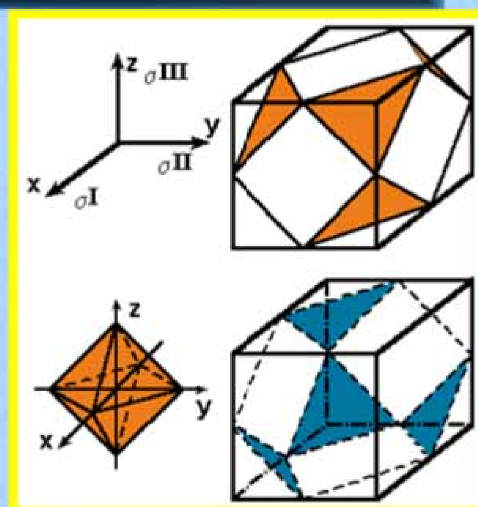
$$\ell = m = n = \pm\frac{1}{\sqrt{3}}$$





Octahedral Planes and Octahedral Stresses

Octahedral stresses are normal and shear stresses acting on the octahedral planes

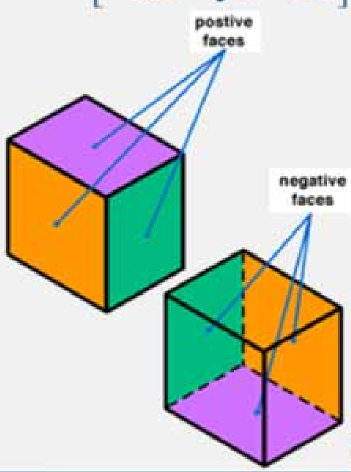


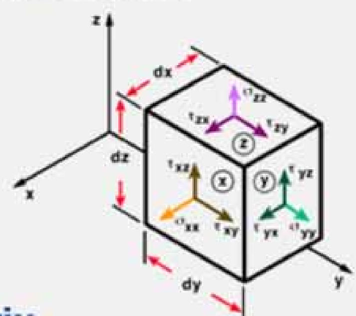
$$\sigma_{oct.} = \frac{1}{3}\left(\sigma^{I} + \sigma^{II} + \sigma^{III}\right) = \frac{1}{3} I_1$$

$$9\tau_{oct.}^2 = \left(\sigma^{I} - \sigma^{II}\right)^2 + \left(\sigma^{II} - \sigma^{III}\right)^2 + \left(\sigma^{III} - \sigma^{I}\right)^2 = 2 I_1^2 - 6 I_2$$

Decomposition of Stress Matrix into Volumetric and Deviatoric Ones

$$\begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{xx}-\frac{1}{3}I_1 & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy}-\frac{1}{3}I_1 & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz}-\frac{1}{3}I_1 \end{bmatrix}}_{\text{deviatoric stress matrix}} + \underbrace{\begin{bmatrix} \frac{1}{3}I_1 & 0 & 0 \\ 0 & \frac{1}{3}I_1 & 0 \\ 0 & 0 & \frac{1}{3}I_1 \end{bmatrix}}_{\text{volumetric stress matrix}}$$





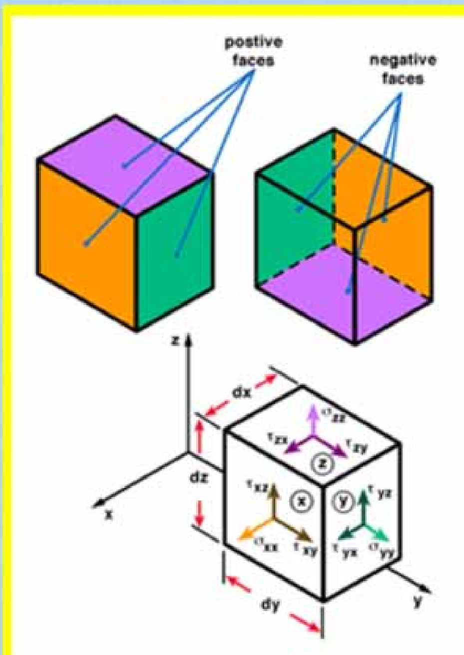
Decomposition of Stress Matrix into Volumetric and Deviatoric Ones

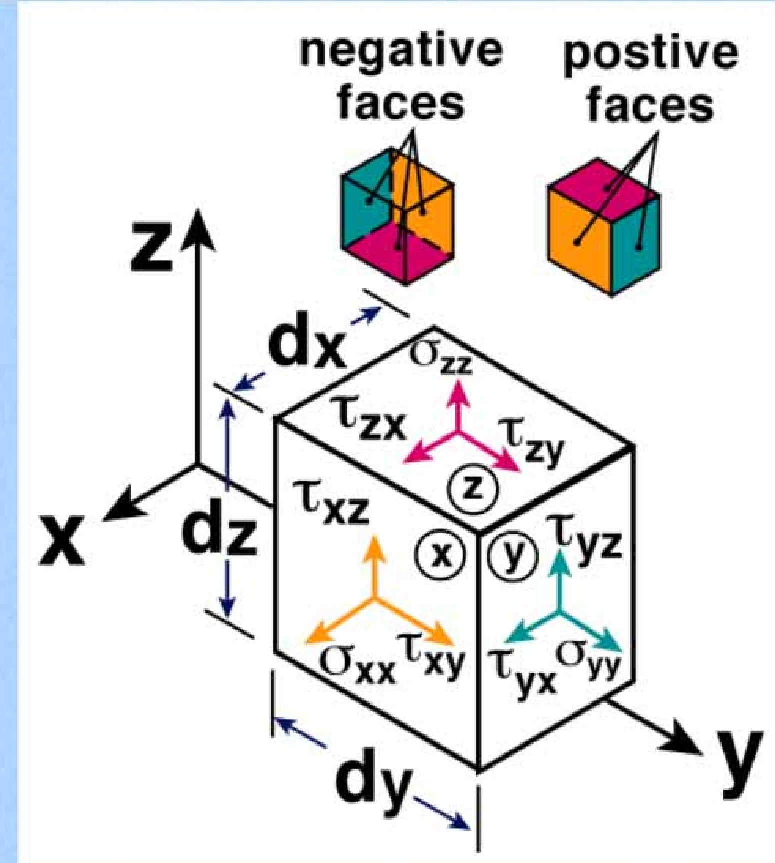
where

$$\boxed{\begin{aligned} I_1 &= \left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) \\ &= \left(\sigma^{I} + \sigma^{II} + \sigma^{III}\right) \end{aligned}}$$

Deviatoric stress components are associated with change in shape.

Volumetric (dilatational) stress components are associated with change in volume.

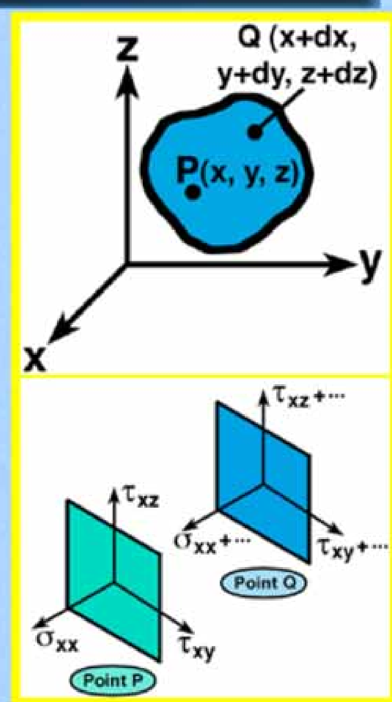




Stresses at Neighboring Points

Point Q is at a distance dx, dy, dz in the x, y, z directions from point P.

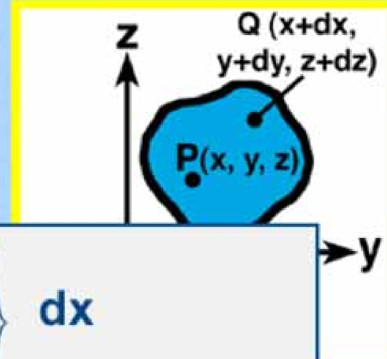
The stress components acting on plane x = const. at point Q are related to those on the parallel plane at point P as follows:



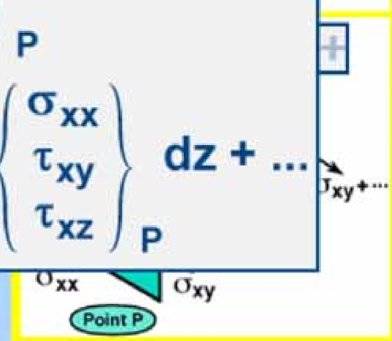
Stresses at Neighboring Points

Point Q is at a distance dx, dy, dz in the x, y, z directions from point P.

The stress components



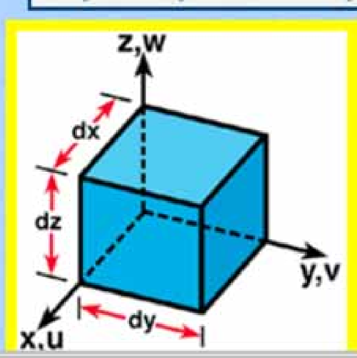
$$
\begin{Bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{Bmatrix}_Q = \begin{Bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{Bmatrix}_P + \frac{\partial}{\partial x}\begin{Bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{Bmatrix}_P dx + \frac{\partial}{\partial y}\begin{Bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{Bmatrix}_P dy + \frac{\partial}{\partial z}\begin{Bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{Bmatrix}_P dz + \ldots
$$

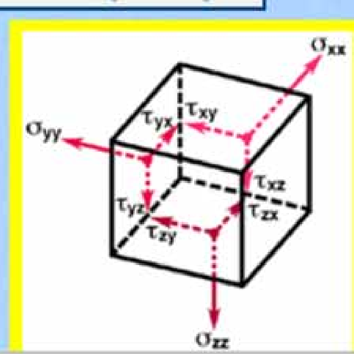


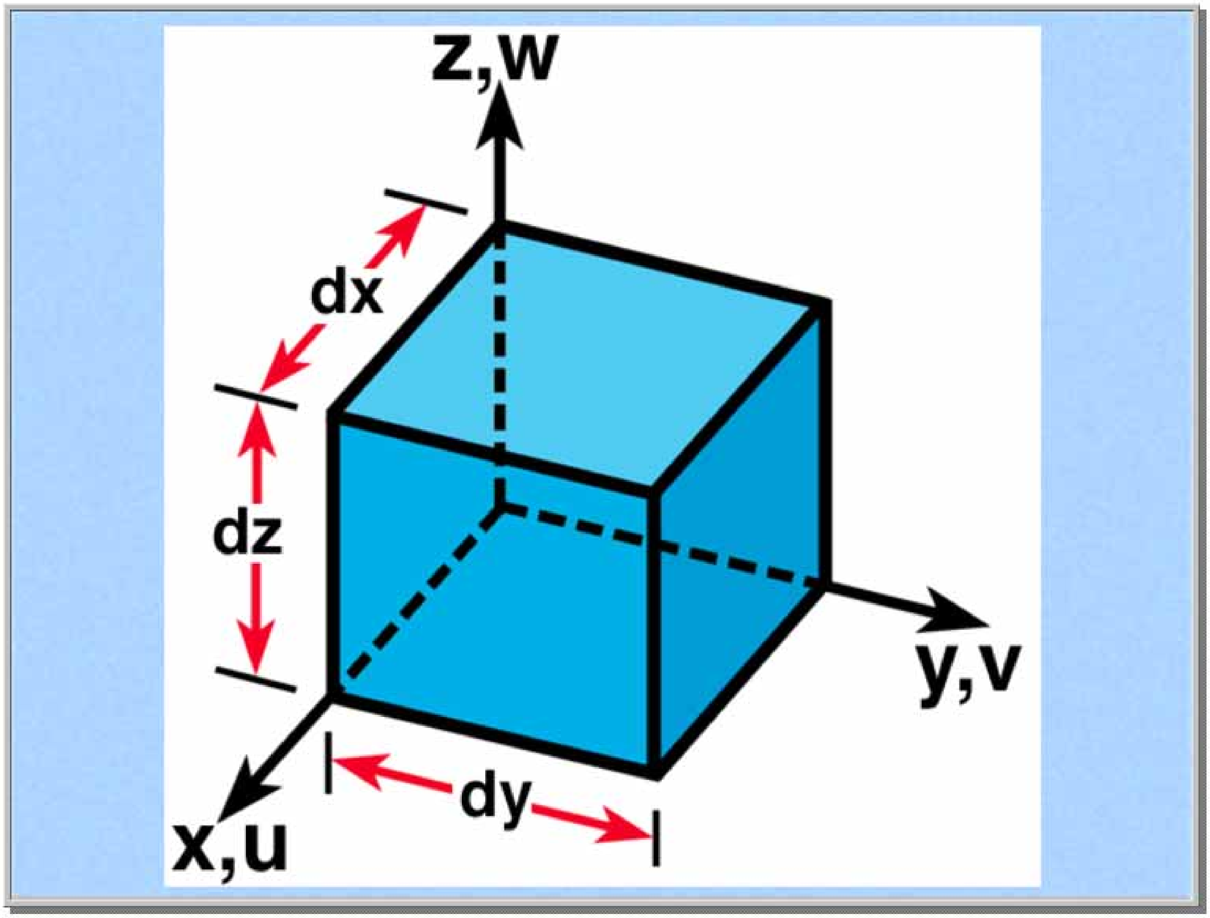
Differential Equations of Motion of a Deformable Body

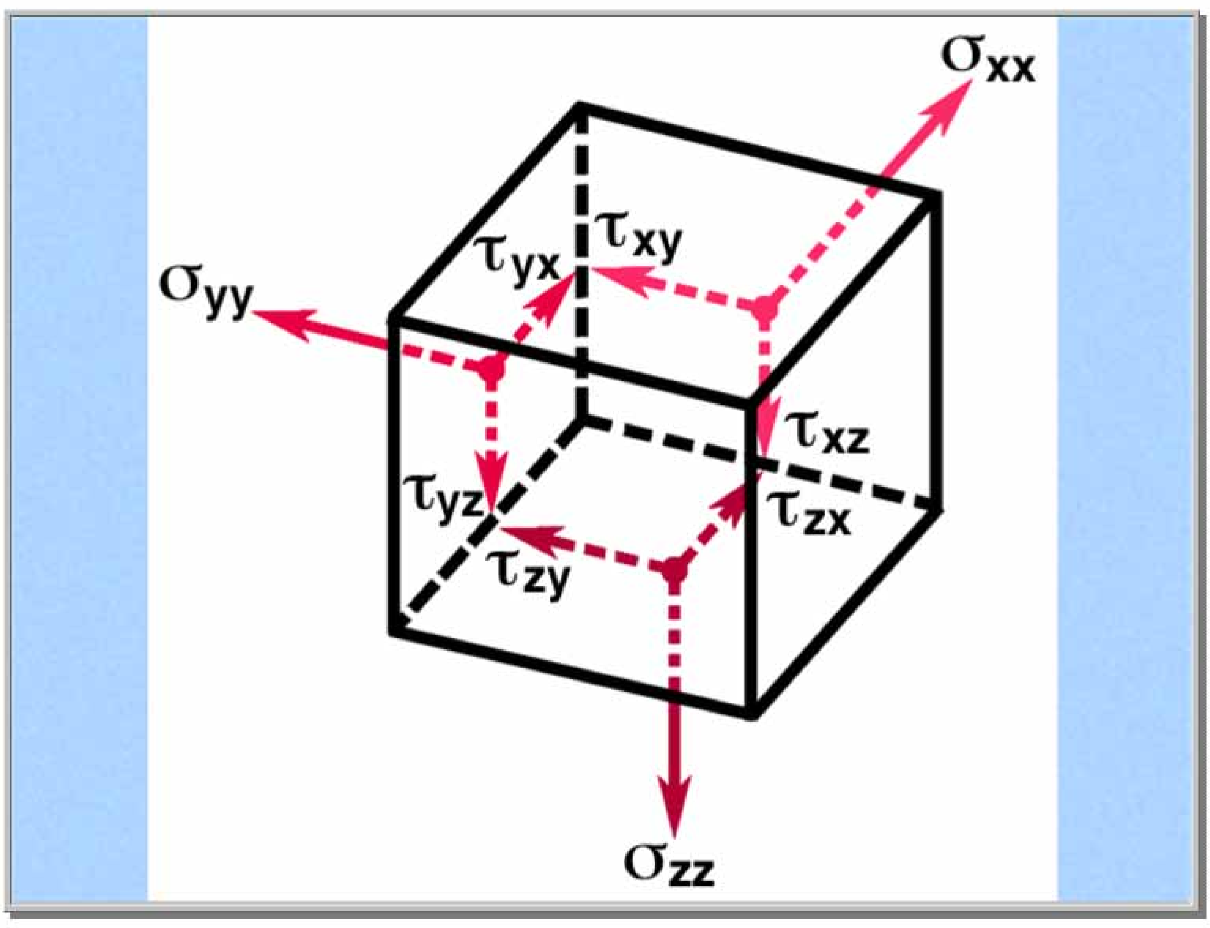
- Consider an infinitesimal element of extent dx, dy, dz in the x, y, z coordinate directions.
- Stress components on the negative faces are:

$$
\begin{Bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{Bmatrix}, \quad \begin{Bmatrix} \tau_{yx} \\ \sigma_{yy} \\ \tau_{yz} \end{Bmatrix}, \quad \begin{Bmatrix} \tau_{zx} \\ \tau_{zy} \\ \sigma_{zz} \end{Bmatrix}
$$






z,w
dx
dz
y,v
dy
x,u


σ_{xx}
τ_{xy}
τ_{yx}
σ_{yy}
τ_{xz}
τ_{yz}
τ_{zx}
τ_{zy}
σ_{zz}

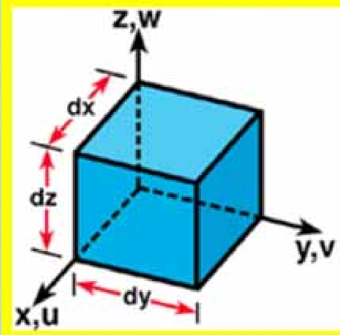
Differential Equations of Motion of a Deformable Body

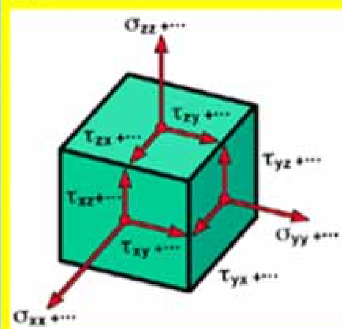
Stress components on the positive faces are:

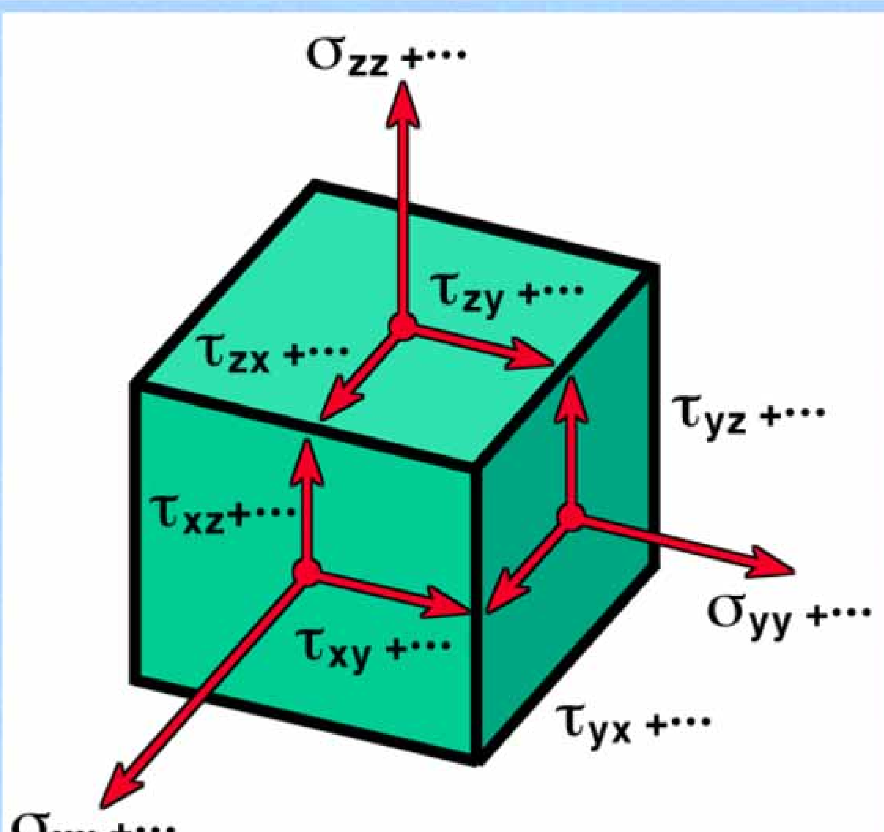
$$\begin{Bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{Bmatrix} + \frac{\partial}{\partial x} \begin{Bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{Bmatrix} dx,$$

$$\begin{Bmatrix} \tau_{yx} \\ \sigma_{yy} \\ \tau_{yz} \end{Bmatrix} + \frac{\partial}{\partial y} \begin{Bmatrix} \tau_{yx} \\ \sigma_{yy} \\ \tau_{yz} \end{Bmatrix} dy,$$

$$\begin{Bmatrix} \tau_{zx} \\ \tau_{zy} \\ \sigma_{zz} \end{Bmatrix} + \frac{\partial}{\partial z} \begin{Bmatrix} \tau_{zx} \\ \tau_{zy} \\ \sigma_{zz} \end{Bmatrix} dz$$



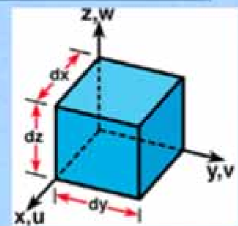


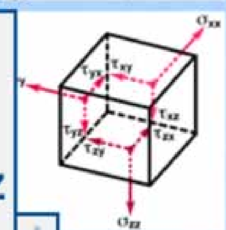


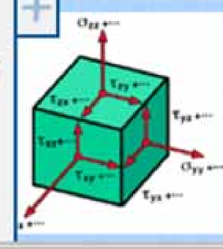
Differential Equations of Motion of a Deformable Body

- Mass of element = $\rho \, dx \, dy \, dz$
  ρ = mass density
- Acceleration in x direction $= \dfrac{\partial^2 u}{\partial t^2}$
- Summing the forces in the x direction

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx\right) dy\, dz - \sigma_{xx}\, dy\, dz$$
$$+ \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx\, dz - \tau_{yx}\, dx\, dz$$
$$+ \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dx\, dy - \tau_{zx}\, dx\, dy$$
$$= \rho\, dx\, dy\, dz \frac{\partial^2 u}{\partial t^2}$$





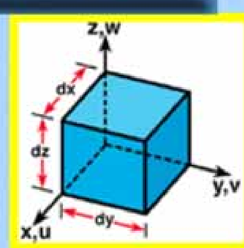


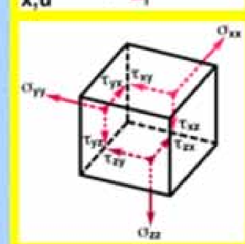
Differential Equations of Motion of a Deformable Body

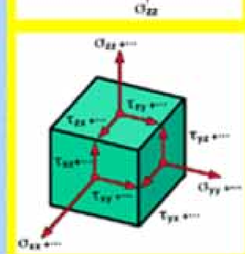
- Mass of element = $\rho \, dx \, dy \, dz$
  ρ = mass density
- Acceleration in x direction $= \dfrac{\partial^2 u}{\partial t^2}$
- Summing the forces in the x direction

or

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2}$$







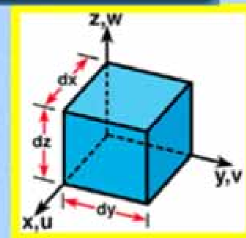
Differential Equations of Motion of a Deformable Body

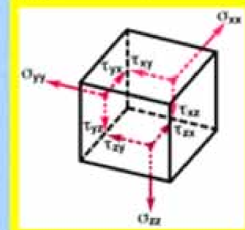
- Summation of forces in the y and z directions leads to:

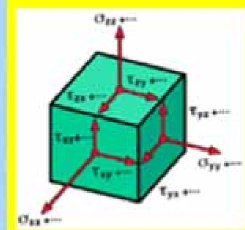
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho \frac{\partial^2 v}{\partial t^2}$$

and

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2}$$







Mohr's Circle Representation

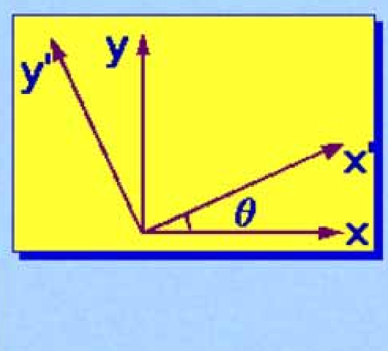
Transformation of Stress Components
Two-Dimensional State of Stress

$$[\sigma'] = [T]^t [\sigma] [T]$$

where $$[\sigma'] = \begin{bmatrix} \sigma_{x'x'} & \tau_{x'y'} \\ \tau_{y'x'} & \sigma_{y'y'} \end{bmatrix}$$

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Mohr's Circle Representation

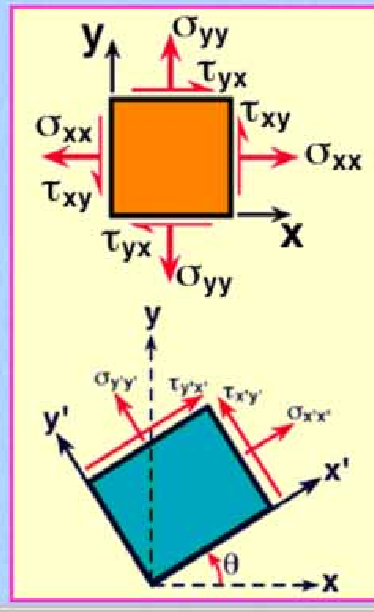
Transformation of Stress Components

$$[\sigma'] = [T]^t [\sigma][T]$$

where $[\sigma'] = \begin{bmatrix} \sigma_{x'x'} & \tau_{x'y'} \\ \tau_{y'x'} & \sigma_{y'y'} \end{bmatrix}$

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

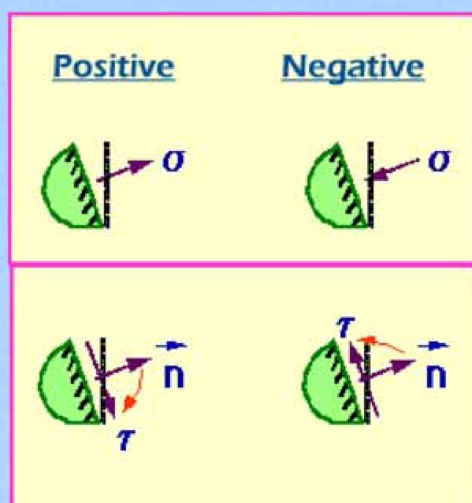


Mohr's Circle Representation

Sign Convention

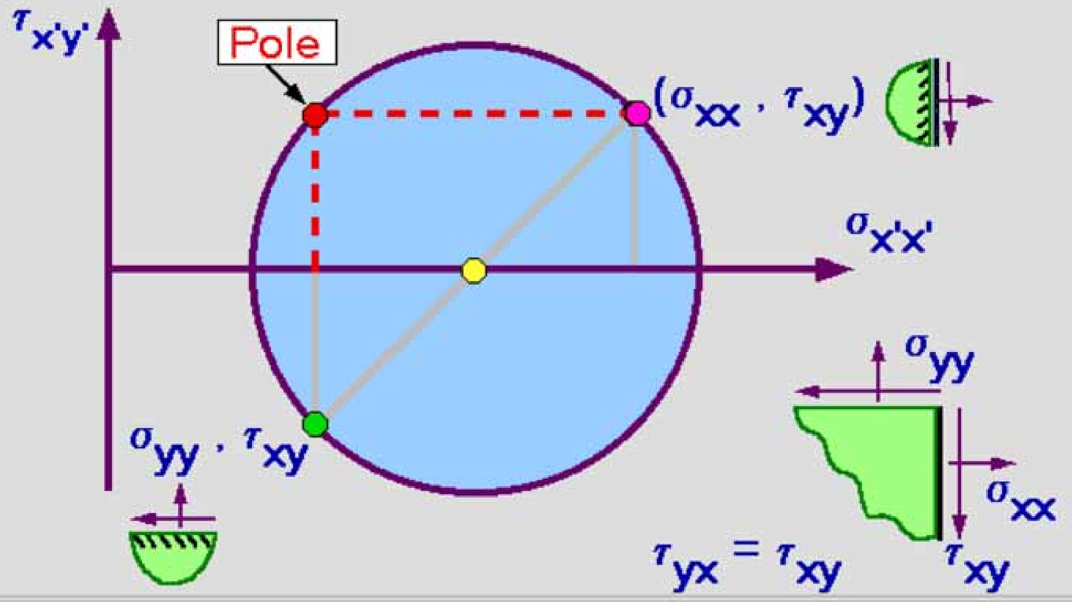
Positive normal stresses are tensile

Positive shear stress clockwise



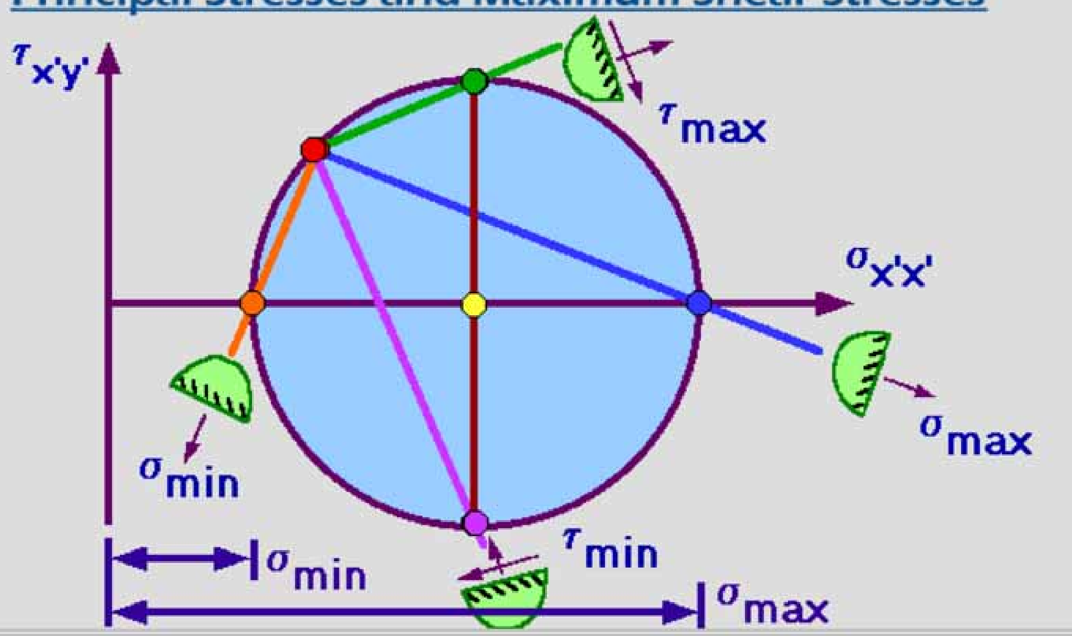
Mohr's Circle Representation

Location of Pole



Mohr's Circle Representation

Principal Stresses and Maximum Shear Stresses



Mohr's Circle Representation

Stresses on any Inclined Plane